Производные

$const' = 0$

$(x^n)' = nx^{n-1}$

$\sqrt{x}' = \dfrac{1}{2\sqrt{x}}$

$\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$

$(a^x)' = a^x \ln a$

$(e^x)' = e^x$

$(\log_a x)' = \dfrac{1}{x \ln a}$

$(\ln x)' = \dfrac{1}{x}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(tg\, x)' = \dfrac{1}{\cos^2 x}$

$(ctg\, x)' = -\dfrac{1}{\sin^2 x}$

$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

$(arctg\, x)' = \dfrac{1}{1+x^2}$

$(arcctg\, x)' = -\dfrac{1}{1+x^2}$

$(sh\, x)' = ch\, x$

$(ch\, x)' = sh\, x$

Дифференциалы

$dx^2 = 2x\, dx$

$dx^n = nx^{n-1}dx$

$d(ax+b) = adx$

$d(\sin x) = \cos x\, dx$

$d(\cos x) = -\sin x\, dx$

$d(tg\, x) = \dfrac{1}{\cos^2 x}dx$

$d(ctg\, x) = -\dfrac{1}{\sin^2 x}$

$d(\ln x) = \dfrac{1}{x}dx$

$d(\log_a x) = \dfrac{1}{x \ln a}dx$

$de^x = e^x dx$

$da^x = a^x \ln a\, dx$

$d\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}dx$

$d\sqrt{x} = \dfrac{1}{2\sqrt{x}}dx$

$r = a, r = 2a\cos\varphi, r = 2a\sin\varphi$ — ок.

$x = a\cos\varphi, y = a\sin\varphi$ — эллипс

$r = a(1 \pm \cos\varphi)$ — кардиоида(ox)

$r = a(1 \pm \sin\varphi)$ — кардиоида(oy)

$r^2 = a^2\cos 2\varphi, r^2 = a^2\cos 2\varphi$

— лемниската Бернулли

$x = a\cos^3\varphi, y = a\sin^3\varphi$ — астроида

$x = a(t - \sin t), y = a(t - \cos t)$

—циклоида

Интегралы

$\int du = u + C$

$\int u^n du = \dfrac{u^{n+1}}{n+1} + C$

$\int \dfrac{du}{u} = \ln|u| + C$

$\int a^u du = \dfrac{a^u}{\ln a} + C$

$\int e^u du = e^u + C$

$\int \sin u\, du = -\cos u + C$

$\int \cos u\, du = \sin u + C$

$\int \dfrac{du}{\sin u} = \ln\left|tg\,\dfrac{u}{2}\right| + C$

$\int \dfrac{du}{\cos u} = \ln\left|tg\left(\dfrac{u}{2} + \dfrac{\pi}{4}\right)\right| + C$

$\int \dfrac{du}{\sin^2 u} = -ctg\, u + C$

$\int \dfrac{du}{\cos^2 u} = tg\, u + C$

$\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$

$\int \dfrac{du}{\sqrt{u^2 + a^2}} = \ln\left|u + \sqrt{u^2 + a^2}\right| + C$

$\int \dfrac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

$\int sh\, u\, du = ch\, u + C$

$\int ch\, u\, du = sh\, u + C$

$\int \dfrac{du}{ch^2 u} = th\, u + C$

$\int \dfrac{du}{sh^2 u} = -cth\, u + C$

$\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a}arctg\,\dfrac{u}{a} + C$

$\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C$

$\int \dfrac{du}{a^2 - u^2} = \dfrac{1}{2a}\ln\left|\dfrac{u+a}{u-a}\right| + C$

$\int \sqrt{a^2 - u^2}\, du = \dfrac{u}{2}\sqrt{a^2 - u^2} + \dfrac{a^2}{2}arcsin\dfrac{u}{a} + C$

$\int \sqrt{u^2 + a^2}\, du = \dfrac{u}{2}\sqrt{u^2 + a^2} + \dfrac{a^2}{2}\ln\left|u + \sqrt{u^2 + a^2}\right| + C$

$\int \sqrt{u^2 - a^2}\, du = \dfrac{u}{2}\sqrt{u^2 - a^2} - \dfrac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

Эквиваленты

$\underline{x \to 0}$

$\sin x \sim x$

$\arcsin x \sim x$

$tg\, x \sim x$

$arctg\, x \sim x$

$\cos x \sim 1 - \dfrac{x^2}{2}$

$\log_a(1+x) \sim \dfrac{x}{\ln a}$

$\ln(1+x) \sim x$

$a^x - 1 \sim x \ln a$

$e^x - 1 \sim x$

$\underline{x \to \infty}$

$\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = e$

Приложения определен. интеграла

$S = \int_a^b f(x)dx$

$S = \int_{t1}^{t2} y(t)x_t'dt$

$S = \dfrac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi$

$V = \int_a^b S(x)dx$

$V_x = \pi\int_a^b f^2(x)dx$

$V_y = \pi\int_c^d g^2(y)dy = 2\pi\int_a^b |x|f(x)dx$

$V_x = \pi\int_{t1}^{t2} y^2(t)x_t'dt$

$V_p = \dfrac{2}{3}\pi\int_\alpha^\beta r^3(\varphi)\sin\varphi\, d\varphi$

$l = \int_a^b \sqrt{1 + (y_x')^2}\, dx$

$l = \int_{t1}^{t2} \sqrt{(x_t')^2 + (y_t')^2}\, dt$

$l = \int_\alpha^\beta \sqrt{r^2 + (r_\varphi')^2}\, d\varphi$

$P_x = 2\pi\int_a^b y(x)\sqrt{1 + (y_x')^2}\, dx$

$P_x = 2\pi\int_{t1}^{t2} y(t)\sqrt{(x_t')^2 + (y_t')^2}\, dt$

$P_p = 2\pi\int_\alpha^\beta r\sin\varphi\sqrt{r^2 + (r_\varphi')^2}\, d\varphi$

Подготовка к РК №1 „Определенный интеграл"

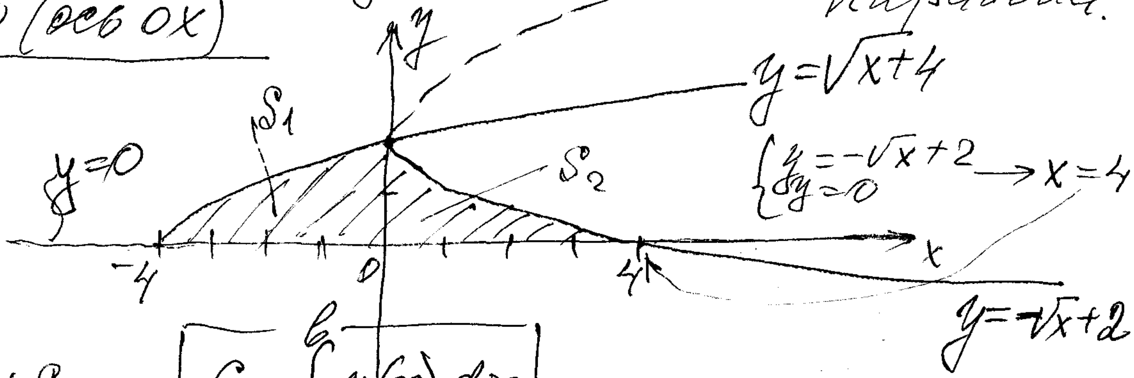
Задачи для подготовки.

1.1 $y=\sqrt{x+4} \longrightarrow y^2=x+4$ парабола

$y=-\sqrt{x}+2 \dashrightarrow y-2=-\sqrt{x} \quad (y-2)^2=x$ парабола.

$y=0$ (ось OX)

$S-?$



$S=S_1+S_2$ $\quad \boxed{S=\int\limits_a^b y(x)\,dx}$

$$S_1=\int\limits_{-4}^{0}\sqrt{x+4}\,dx=\frac{(x+4)^{3/2}}{\frac{3}{2}}\Bigg|_{-4}^{0}=\frac{2}{3}\left(4^{3/2}-0\right)=\frac{2}{3}\cdot 8=\frac{16}{3}$$

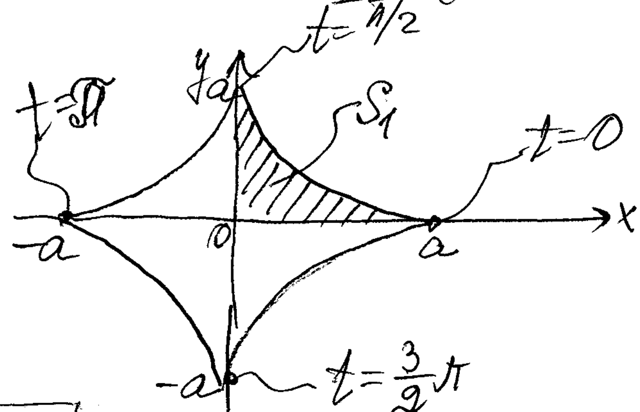
$$S_2=\int\limits_0^4\left(2-\sqrt{x}\right)dx=2x\Big|_0^4-\frac{x^{3/2}}{3/2}\Bigg|_0^4=2(4-0)-\frac{2}{3}\left(4^{3/2}-0\right)=$$

$$=8-\frac{2}{3}\cdot 8=\left(8-\frac{16}{3}\right)$$

$$S=S_1+S_2=\frac{16}{3}+\left(8-\frac{16}{3}\right)=\underline{\underline{8\text{ ед}^2}}$$

1.2 $\begin{cases} x=a\cos^3 t \\ y=a\sin^3 t \end{cases}$

$S-?$



$S=4S_1$
(симметрия)

$$\boxed{S=\int\limits_{t_1}^{t_2} y(t)\,\underbrace{x'_t(t)\,dt}_{dx}}$$

$x=a\cos^3 t \to dx=-3a\cos^2 t\sin t\,dt$

$$S=4\int\limits_{\pi/2}^{0} a\sin^3 t\left(-3a\cos^2 t\sin t\,dt\right)=-12a^2\int\limits_{\pi/2}^{0}\sin^4 t\cos^2 t\,dt$$

$$= -12a^2\int_{\pi/2}^{0}\frac{(1-\cos 2t)^2}{4}\,\frac{(1+\cos 2t)}{2}\,dt =$$

$$= -\frac{12a^2}{8}\int_{\pi/2}^{0}\left(1-2\cos 2t+\cos^2 2t+\cos 2t-2\cos^2 2t+\cos^3 2t\right)dt =$$

$$= -\frac{3}{2}a^2\int_{\pi/2}^{0}\left(1-\cos 2t-\cos^2 2t+\cos^3 2t\right)dt =$$

$$= -\frac{3}{2}a^2 t\Big|_{\pi/2}^{0} + \frac{3}{2}a^2\int_{\pi/2}^{0}\cos 2t\,dt + \frac{3}{2}a^2\int_{\pi/2}^{0}\frac{1+\cos 4t}{2}\,dt -$$

$$-\frac{3}{2}a^2\int_{\pi/2}^{0}\cos^2 2t\,\cos 2t\,dt =$$

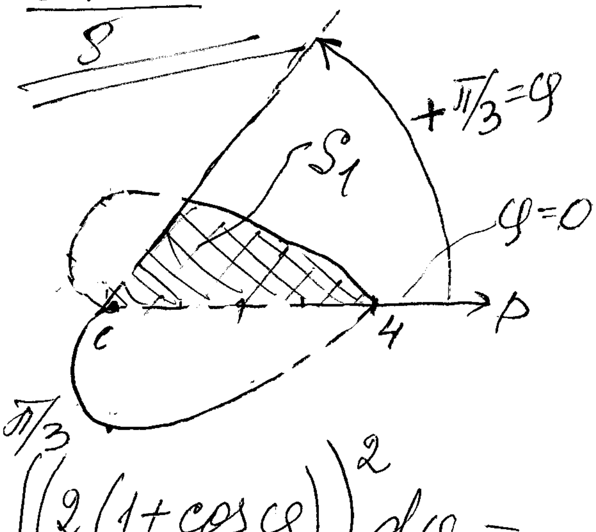
$$= -\frac{3}{2}a^2\left(0-\frac{\pi}{2}\right) + \frac{3a^2}{4}\underset{\to 0}{\sin 2t}\Big|_{\pi/2}^{0} + \frac{3}{4}a^2 t\Big|_{\pi/2}^{0} + \frac{3}{4}a^2\int_{\pi/2}^{0}\cos 4t\,dt$$

$$-\frac{3}{2\cdot 2}a^2\int_{\pi/2}^{0}\left(1-\sin^2 2t\right)d\sin 2t =$$

$$= \frac{3}{2}\frac{\pi a^2}{2} + 0 + \frac{3}{4}a^2\left(0-\frac{\pi}{2}\right) + \frac{3a^2}{4\cdot 4}\underset{\to 0}{\sin 4t}\Big|_{\pi/2}^{0} -$$

$$-\frac{3a^2}{4}\underset{\to 0}{\sin 2t}\Big|_{\pi/2}^{0} + \frac{3}{4}a^2\,\frac{\underset{\to 0}{\sin^3 2t}}{3}\Big|_{\pi/2}^{0} =$$

$$= \frac{3\pi a^2}{4} - \frac{3\pi a^2}{8} + 0 - 0 + 0 = \frac{3\pi a^2}{8}$$



1.3] $\begin{cases}\rho = 2(1+\cos\varphi)\\ \varphi = 0 \quad \varphi = \frac{\pi}{3}\end{cases}$

$S - ?$

$$\boxed{S = \frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\varphi)\,d\varphi}$$

$$S = \frac{1}{2}\int_{0}^{\pi/3}\left(2(1+\cos\varphi)\right)^2 d\varphi =$$

$$= \frac{1}{2}\cdot 4\left(\int_{0}^{\pi/3}\left(1+2\cos\varphi+\cos^2\varphi\right)d\varphi\right) =$$

$$=2\left(\int_0^{\pi/3} d\varphi + 2\int_0^{\pi/3}\cos\varphi\, d\varphi + \int_0^{\pi/3}\frac{1+\cos 2\varphi}{2}\, d\varphi\right) =$$ ③

$$=2\varphi\Big|_0^{\pi/3} + 4\sin\varphi\Big|_0^{\pi/3} + \cancel{2}\cdot\frac{1}{2}\int_0^{\pi/3} d\varphi + \cancel{2}\frac{1}{2}\int_0^{\pi/3}\cos 2\varphi\, d\varphi =$$

$$=2\left(\frac{\pi}{3}-0\right)+4\left(\sin\frac{\pi}{3}-\sin 0\right)+\varphi\Big|_0^{\pi/3}+\frac{1}{2}\sin 2\varphi\Big|_0^{\pi/3}=$$

$$=\frac{2}{3}\pi+4\cdot\frac{\sqrt3}{2}+\left(\frac{\pi}{3}-0\right)+\frac{1}{2}\left(\sin\frac{2}{3}\pi-\sin 0\right)=$$

$$=\frac{2}{3}\pi+2\sqrt3+\frac{\pi}{3}+\frac{1}{2}\frac{\sqrt3}{2}=\pi+2\sqrt3+\frac{\sqrt3}{4}=$$

$$=\underline{\underline{\frac{4\pi+9\sqrt3}{4}\ \text{ед}^2}}$$

$\boxed{2.1}$ см. N6.538 (семинар №8)

$\boxed{2.2}$ $\begin{cases} x=at^2 \\ y=a\ln t \\ a>0 \end{cases}$ ООЗ $t>0$.

$x=0$ $y=0$

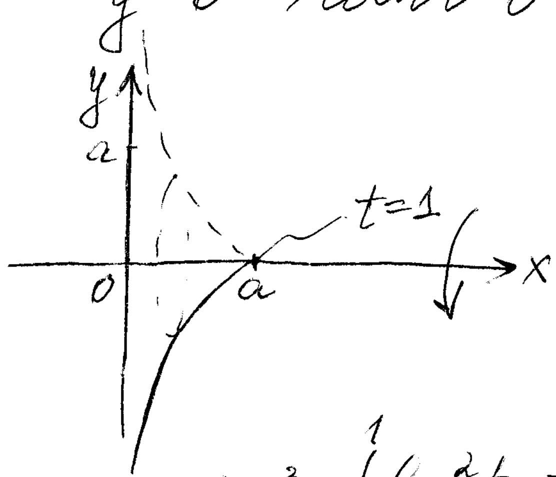
$x=0 \to at^2=0 \to t=0$

$y=0 \to a\ln t=0 \quad \ln t=0$

$t=1$

$V_x - ?$

$$\boxed{V_x=\pi\int_{t_1}^{t_2} y^2(t)\, x'_t\, dt}$$



$$V_x=\pi\int_0^1 a^2\ln^2 t\; 2at\, dt = 2a^3\pi\int_0^1 \ln^2 t\cdot t\, dt =$$

$$\int t\ln^2 t\, dt = \left|\begin{matrix} u=\ln^2 t & du=2\ln t\,\frac{1}{t}dt \\ dv=t\,dt & v=\int t\,dt=\frac{t^2}{2}\end{matrix}\right| =$$

$$=\frac{t^2}{2}\ln^2 t-\int\frac{t^2}{2}\frac{2}{t}\ln t\, dt=\frac{t^2}{2}\ln^2 t-\int t\ln t\, dt=$$

$$=\left|\begin{matrix} u=\ln t & du=\frac{dt}{t} \\ dv=t\,dt \to & v=\int t\,dt=\frac{t^2}{2}\end{matrix}\right| =$$

$$= \frac{t^2}{2}\ln^2 t - \left(\frac{t^2}{2}\ln t - \int \frac{t^2}{2}\frac{dt}{t}\right) =$$

(4)

$$= \frac{t^2}{2}\ln^2 t - \frac{t^2}{2}\ln t + \frac{1}{2}\int t\,dt =$$

$$= \frac{t^2}{2}\ln^2 t - \frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$\boxed{=}\ 2\pi a^3\left(\frac{t^2}{2}\ln^2 t - \frac{t^2}{2}\ln t + \frac{t^2}{4}\right)\Bigg|_0^1 =$$

$$= 2\pi a^3\left[\left(\frac{1}{2}\underbrace{\ln^2 1}_{\to 0} - \frac{1}{2}\underbrace{\ln 1}_{\to 0} + \frac{1}{4}\right) - \left(0\ln^2 0 - 0\ln 0 + 0\right)\right] \boxed{=}$$

$$\lim_{t\to 0+}\frac{t^2}{2}\ln^2 t = [0\cdot\infty] = \frac{1}{2}\lim_{t\to 0}\frac{\ln^2 t}{\frac{2}{t^2}}\left[\frac{\infty}{\infty}\right] \overset{\text{Б-Л.}}{=} \lim_{t\to 0}\frac{2\ln t\,\frac{1}{t}}{\left(-\frac{4}{t^3}\right)} =$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{\ln t}{\frac{1}{t^2}}\left[\frac{\infty}{\infty}\right] = -\frac{1}{2}\lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{2}{t^3}} = \frac{1}{4}\lim t^2 = 0$$

Аналогично $\lim\limits_{t\to 0}\frac{t^2}{2}\ln t = 0.$

$$\boxed{=}\ 2\pi a^3\cdot\frac{1}{4} = \frac{\pi a^3}{2}$$

2.3 $r = a\sin^2\varphi$

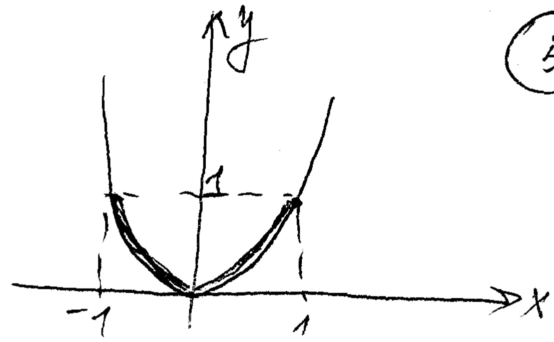
$V_p - ?$

см. N 6.543 (семинар 8)

5

3.1 $y=x^2 \quad A(-1;1)$
$\qquad\qquad\;\; B(1,1)$

$l_{AB}-?$



$$\boxed{l=\int_a^b\sqrt{1+(y'_x)^2}\,dx}$$ длина дуги

$y=x^2 \quad y'=2x$

$$l=2\int_0^1\sqrt{1+(2x)^2}\,dx=2\int_0^1\sqrt{1+4x^2}\,dx=$$
симметрия

$$\boxed{\int\sqrt{a^2+u^2}\,du=\frac{u}{2}\sqrt{u^2+a^2}+\frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right|+c}$$

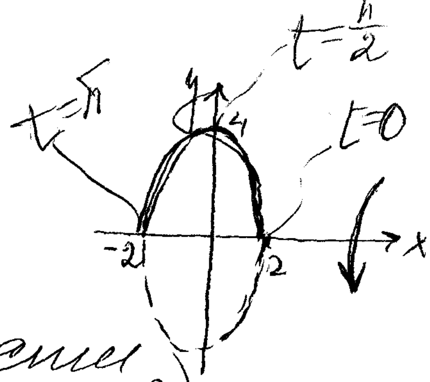
$$=\cancel{2}\cdot\frac{1}{\cancel{2}}\int_0^1\sqrt{1+(2x)^2}\,d(2x)=$$

$$=\left(\frac{\cancel{2}x}{\cancel{2}}\sqrt{1+4x^2}+\frac{1}{2}\ln\left|2x+\sqrt{1+4x^2}\right|\right)\Bigg|_0^1=$$

$$=1\cdot\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{2})-\left(0+\frac{1}{2}\ln(0+1)\right)=$$

$$=\underline{\sqrt{2}+\frac{1}{2}\ln(2+\sqrt{2})}$$

3.2 $\begin{cases}x=2\cos t\\ y=4\sin t\end{cases}$ — эллипс $\begin{matrix}a=2\\ b=4\end{matrix}$



$Q_x-?$ (площадь поверхности вращения)

$$\boxed{Q_x=2\pi\int_{t_1}^{t_2}y(t)\sqrt{(x'_t)^2+(y'_t)^2}\,dt}$$ в параметрическом виде

$x'_t=-2\sin t \quad y'_t=4\cos t$

$$Q_x=2\cdot2\pi\int_0^{\pi/2}4\sin t\sqrt{4\sin^2t+16\cos^2t}\,dt=$$
симметрия

$$= 16\pi \int_0^{\pi/2} \sin t \sqrt{4(\sin^2 t + 4\cos^2 t)}\, dt =$$

$$= 32\pi \int_0^{\pi/2} \sin t \sqrt{1-\cos^2 t + 4\cos^2 t}\, dt =$$

$$= -32\pi \int_0^{\pi/2} \sqrt{1+3\cos^2 t}\, d\cos t =$$

$$= -\frac{32\pi}{\sqrt{3}} \int_0^{\pi/2} \sqrt{1+(\sqrt{3}\cos t)^2}\, d(\sqrt{3}\cos t) =$$

$$= -\frac{32\pi}{\sqrt{3}} \left(\frac{\sqrt{3}\cos t}{2}\sqrt{1+3\cos^2 t} + \frac{1}{2}\ln\left|\sqrt{3}\cos t + \sqrt{1+3\cos^2 t}\right| \right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= -\frac{32\pi}{\sqrt{3}}\left[\underbrace{\left(\frac{\sqrt{3}}{2}\cdot 0\cdot 1 + \frac{1}{2}\ln(0+1)\right)}_{\to 0} - \left(\frac{\sqrt{3}}{\cancel{2}}\cdot \cancel{2} + \frac{1}{2}\ln(\sqrt{3}+2)\right)\right] =$$

$$= \frac{32\pi}{\sqrt{3}}\left(\sqrt{3} + \ln\sqrt{2+\sqrt{3}}\right)$$

$\boxed{4.1}$ Исследовать на сходимость

$$\int_1^{+\infty} \frac{\operatorname{arctg}\sqrt{1+x^2}\, dx}{x+3}$$ Н.И. $1^{\text{ого}}$ рода

$x \to +\infty$ $$f(x) = \frac{\operatorname{arctg}\sqrt{1+x^2}}{x+3} = \frac{\overbrace{\operatorname{arctg}\sqrt{1+x^2}}^{\to \frac{\pi}{2}}}{x\left(1+\underbrace{\frac{3}{x}}_{\to 0}\right)} \sim$$

$$\sim \frac{\pi}{2x} = g(x)$$

const не влияет на сходимость

$$\frac{\pi}{2}\int_1^{+\infty}\frac{dx}{x}$$ — расходится как интеграл Дирихле $(p=1)$

$f(x) \sim g(x) \to$ интегралы ведут себя одинаково

Ответ: исходный расходится по предельному признаку

$\boxed{4.2}$ $\displaystyle\int_1^{+\infty} \frac{x^2+x+1}{x^4+\cos x}\,dx$ Н.И $1^{ого}$ рода (7)

$x\to+\infty$ $f(x)=\dfrac{x^2+x+1}{x^4+\cos x}=\dfrac{x^2(1+\frac{1}{x}+\frac{1}{x^2})}{x^4(1+\frac{\cos x}{x^4})}\sim$

$\sim\dfrac{1}{x^2}=g(x)$ $\dfrac{\cos x}{x^4}\to 0$ $\left[\dfrac{\text{огранич}}{\infty}\right]$

$\displaystyle\int_1^{+\infty}\frac{dx}{x^2}$ – сходится как интеграл Дирихле

Но $f(x)\sim g(x)\to$ исходный сходится по предельному признаку

$\boxed{4.3}$ а) $\displaystyle\int_1^{\infty}\frac{\sin x}{x^{4/3}}\,dx$ Н.И $1^{ого}$ рода

$f(x)=\dfrac{\sin x}{x^{4/3}}$ знакопеременная на $[1;+\infty)\to$

Оценим $|f(x)|=\dfrac{|\sin x|}{x^{4/3}}\le\dfrac{1}{x^{4/3}}=g(x)$.

сх. $\leftarrow$ сх.

сходится по признаку сравнения $\leftarrow$ $\displaystyle\int_1^{+\infty}\frac{dx}{x^{4/3}}$ – сходится $\left(p=\frac{4}{3}>1\right)$

А т.к. $\displaystyle\int_1^{\infty}|f(x)|\,dx$ сходится, то исходный сх. абсолютно

б) $\displaystyle\int_0^1\frac{\sin x\,dx}{x^{4/3}}$ Н.И. $2^{ого}$ рода

$x=0$ – т.р.

0 1

$x\to0+$

$f(x)=\dfrac{\sin x}{x^{4/3}}\sim\dfrac{x}{x^{4/3}}\sim\dfrac{1}{x^{1/3}}=g(x)$

$\displaystyle\int_0^1\frac{dx}{x^{1/3}}$ – сходится $\left(p=\frac{1}{3}<1\right)$ интеграл Дирихле

$f(x)\sim g(x)\Rightarrow$ исходный сходится по предельн. признаку.

$\boxed{4.4}$ $\int_1^{+\infty} \frac{2+\cos x}{x\sqrt{x}+3}\,dx$ Н.И. 1-го рода

(8)

$|\cos x| \le 1$

при $x \to +\infty$ $2+\cos x = const \in [1;3]$

$$f(x) = \frac{2+\cos x}{x\sqrt{x}+3} = \frac{2+\cos x}{x^{3/2}\left(1+\underbrace{\frac{3}{x^{3/2}}}_{\to 0}\right)} \underset{x\to+\infty}{\sim} \frac{const}{x^{3/2}} = g(x)$$

$f(x) \sim g(x)$

$const\int_1^{+\infty} \frac{dx}{x^{3/2}}$ — сходящая

$p = \frac{3}{2} > 1$

исходный сходящая по предельному признаку.