Семинар 2 по интегралам

Интегрирование по частям

$$\int u\,dv = uv - \int v\,du$$

I тип. $$\int \underbrace{P_n(x)}_{u} \underbrace{\begin{bmatrix} \cos Q_m(x) \\ \sin Q_m(x) \\ e^{Q_m(x)} \\ a^{Q_m(x)} \end{bmatrix} dx}_{dv}$$, где $P_n(x), Q_m(x)$ многочлены степеней n и m от x.

№ 1214.

$$\int \underbrace{x}_{u}\, \underbrace{\sin x\, dx}_{dv} = \left[\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin x\, dx \Rightarrow v = \int dv = \int \sin x\, dx = \\ \qquad = -\cos x + C \end{array}\right]$$

не берём

$$= \underbrace{x}_{u} \cdot \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\, \underbrace{dx}_{du} =$$

$$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$$

Краткая запись:

$$\int x \sin x\, dx = \int x\, d(-\cos x) =$$

$$= x(-\cos x) - \int (-\cos x)\, dx = -x\cos x + \int \cos x\, dx =$$

$$= -x\cos x + \sin x + C$$

№1219.

$$\int \underbrace{(x^2-2x+5)}_{u}\underbrace{e^{-x}dx}_{dv} =$$

$$= \left[\begin{array}{ll} u = x^2-2x+5 & \Rightarrow du = u'dx = (2x-2)\,dx \\ dv = e^{-x}dx & \Rightarrow v = \int dv = \int e^{-x}dx = -\int e^{-x}d(-x) = \\ & = -e^{-x} + \underset{\nwarrow \text{не берём}}{C} \end{array}\right] =$$

$$= \underbrace{(x^2-2x+5)}_{u}\underbrace{(-e^{-x})}_{v} - \int(-e^{-x})(2x-2)\,dx =$$

$$= -(x^2-2x+5)e^{-x} + 2\int \underbrace{(x-1)}_{a}\underbrace{e^{-x}dx}_{d\theta} =$$

$$= \left[\begin{array}{l} a = x-1 \Rightarrow da = a'dx = 1dx = dx \\ d\theta = e^{-x}dx \underset{\uparrow \text{кратко}}{=} -e^{-x}d(-x) = -de^{-x} = d(-e^{-x}) \\ \qquad\qquad\qquad\qquad \theta = -e^{-x} \end{array}\right] =$$

$$= -(x^2-2x+5)e^{-x} + 2\left(\underbrace{(x-1)}_{a}\underbrace{(-e^{-x})}_{\theta} - \int \underbrace{(-e^{-x})}_{\theta}\underbrace{dx}_{da}\right) =$$

$$= -(x^2-2x+5)e^{-x} - 2(x-1)e^{-x} + 2\int e^{-x}d(x) =$$

$$= -(x^2-2x+5)e^{-x} - (2x-2)e^{-x} - 2e^{-x} + C =$$

$$= -(x^2 \cancel{-2x} + 5 + \cancel{2x} \cancel{-2} \cancel{+2})e^{-x} + C = -(x^2+5)e^{-x} + C$$

$\boxed{\text{Д/З I №1215, 1222, 1217}}$

II тип

$$\int \underbrace{\begin{bmatrix} \arcsin Q_m(x) \\ \arccos Q_m(x) \\ \operatorname{arctg} Q_m(x) \\ \operatorname{arcctg} Q_m(x) \\ \ln Q_m(x) \\ \log_a Q_m(x) \end{bmatrix}}_{u} \underbrace{P_n(x)\,dx}_{dv}$$

№1211.

$$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = \left[\begin{array}{l} u=\ln x \Rightarrow du=\frac{1}{x}\,dx \\ dx=dv \Rightarrow x=v \quad \text{пояснение. (это одно из } v\text{)} \end{array}\right] =$$

$$= \underbrace{\ln x}_{u} \cdot \underbrace{x}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x}\,dx}_{du} = x\ln x - \int dx =$$

$$= x\ln x - x + C$$

№1213

$$\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv} = \left[\begin{array}{l} u=\arcsin x \Rightarrow du=\frac{1}{\sqrt{1-x^2}}\,dx \\ dv=dx \Rightarrow v=x \end{array}\right] =$$

$$= \underbrace{\arcsin x}_{u} \cdot \underbrace{x}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}\,dx}_{du} = \left[x\,dx = \frac{1}{2}\,d(x^2)\right]$$

(Кратко: $\int \arcsin x\,dx = \arcsin x \cdot x - \int x\,d\arcsin x =$
$= x\arcsin x - \int x\,\frac{1}{\sqrt{1-x^2}}\,dx =$ и т.д.)

$$= x\arcsin x - \frac{1}{2}\int \frac{d(x^2)}{\sqrt{1-x^2}} = \left[d(x^2) = d(x^2-1) = -d(1-x^2) \right] =$$

$$= x\arcsin x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(1-x^2) =$$

$$= x\arcsin x + \frac{\cancel{1}}{\cancel{2}} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{\cancel{1}}{\cancel{2}}} + C =$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

№1223.

$$\int x^2 \ln x dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{x^2 dx}_{dv} =$$

$$= \left[\begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x} dx \\ dv = x^2 dx = \frac{1}{3} d(x^3) = d(\frac{1}{3}x^3) \Rightarrow v = \frac{1}{3}x^3 \end{array} \right] =$$

$$= \underbrace{\ln x}_{u} \cdot \underbrace{\frac{1}{3}x^3}_{v} - \int \underbrace{\frac{1}{3}x^3}_{v} \cdot \underbrace{\frac{1}{x} dx}_{du} =$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C =$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

(Кратко: $\int \ln x \, x^2 dx = \int \ln x \, d(\frac{x^3}{3}) = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} d(\ln x) =$
$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \cdot \frac{1}{x} dx =$ и т.д.)

№1227

запишем кратко:

$$\int x\,\mathrm{arctg}\,x\,dx = \int \underbrace{\mathrm{arctg}\,x}_{u}\cdot\underbrace{x\,dx}_{dv} = \left[x\,dx = d\left(\frac{x^2}{2}\right)\right] =$$

$$= \int \underbrace{\mathrm{arctg}\,x}_{u}\,\underbrace{d\left(\frac{x^2}{2}\right)}_{dv} = \underbrace{\mathrm{arctg}\,x}_{u}\cdot\underbrace{\frac{x^2}{2}}_{v} - \int \frac{x^2}{2}\,d(\mathrm{arctg}\,x) =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int x^2\frac{1}{1+x^2}\,dx =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\boxed{\int \frac{1+x^2-1}{1+x^2}\,dx} =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{1+x^2} =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}x + \frac{1}{2}\mathrm{arctg}\,x + C$$

Д/З II:

№1212

№1226* Указ. $\frac{1}{\sqrt{x}}dx = x^{-\frac{1}{2}}dx = d\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) = d(2\sqrt{x})$

№1229* Указ. $\int \underbrace{\ln(x+\sqrt{1+x^2})}_{u}\,\underbrace{dx}_{dv}$

№1231* Указ. $\frac{\cos x}{\sin^2 x}dx = \frac{d\sin x}{\sin^2 x} = \sin^{-2}x\,dx = d\,\frac{\sin^{-1}x}{-1} = d\left(-\frac{1}{\sin x}\right) = dv$

III тип („смешанный").

Например, $\int \underbrace{e^{ax}}_{u} \underbrace{\begin{bmatrix} \sin bx \\ \cos bx \end{bmatrix} dx}_{dv} \overset{\text{или}}{=}$

$\overset{\text{или}}{=} \int \underbrace{\begin{bmatrix} \sin bx \\ \cos bx \end{bmatrix}}_{u} \underbrace{e^{ax} dx}_{dv}$

При решении получим уравнение относит. исходного интеграла.

≈ №1234

$$I = \int \underbrace{e^{3x}}_{u} \underbrace{\sin 2x\, dx}_{dv} =$$

$$= \left[\begin{array}{l} u = e^{3x} \Rightarrow du = 3e^{3x} dx \\ dv = \sin 2x\, dx = \frac{1}{2} \sin 2x\, d(2x) = \frac{1}{2} d(-\cos 2x) = \\ \quad = d(-\frac{1}{2}\cos 2x) \Rightarrow \quad v = -\frac{1}{2}\cos 2x \end{array} \right] =$$

$$= \underbrace{e^{3x}}_{u} \underbrace{\left(-\frac{1}{2}\cos 2x\right)}_{v} - \int \underbrace{\left(-\frac{1}{2}\cos 2x\right)}_{v} \cdot \underbrace{3e^{3x} dx}_{du} =$$

$$= -\frac{1}{2} e^{3x} \cos 2x + \frac{3}{2} \int \underbrace{e^{3x}}_{a} \underbrace{\cos 2x\, dx}_{db} =$$

$$= \left[\begin{array}{l} a = e^{3x} \Rightarrow da = 3e^{3x} dx \\ db = \cos 2x\, dx = \frac{1}{2} \cos 2x\, d(2x) = \frac{1}{2} d(\sin 2x) = \\ \quad = d(\frac{1}{2} \sin 2x) \Rightarrow b = \frac{1}{2} \sin 2x \end{array} \right] =$$

$$=-\frac{1}{2}e^{3x}\cos 2x+\frac{3}{2}\left(\underbrace{e^{3x}}_{a}\cdot\underbrace{\frac{1}{2}\sin 2x}_{b}-\int\underbrace{\frac{1}{2}\sin 2x}_{b}\cdot\underbrace{3e^{3x}dx}_{da}\right)=$$

$$=-\frac{1}{2}e^{3x}\cos 2x+\frac{3}{4}e^{3x}\sin 2x-\frac{9}{4}\int e^{3x}\sin 2x\,dx=$$

$$=\frac{1}{4}e^{3x}(3\sin 2x-2\cos 2x)-\frac{9}{4}I.$$

Получим ур-е относит. I:

$$I=\frac{1}{4}e^{3x}(3\sin 2x-2\cos 2x)-\frac{9}{4}I \quad |\cdot 4$$

$$4I=e^{3x}(3\sin 2x-2\cos 2x)-9I$$

$$13I=e^{3x}(3\sin 2x-2\cos 2x)+C$$

$$I=\frac{1}{13}e^{3x}(3\sin 2x-2\cos 2x)+\underline{\underline{C}},\ C\in\mathbb{R}$$

$\boxed{\text{Д13 } \overline{\underline{\text{III}}}\ \ N\ 1232}$

Мы переобозначили $\frac{C}{13}$ через C.

№1252

Д/З IV №1253

Получить табл. инт-л. См →

$$\boxed{I = \int \underbrace{\sqrt{a^2-x^2}}_{u}\,\underbrace{dx}_{dv} =}$$

$$= \underbrace{\sqrt{a^2-x^2}}_{u} \cdot \underbrace{x}_{v} - \int \underbrace{x}_{v}\, \underbrace{d\sqrt{a^2-x^2}}_{du} =$$

$$= x\sqrt{a^2-x^2} - \int x \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}\,dx =$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx =$$

$$= x\sqrt{a^2-x^2} - \int \frac{-a^2+a^2-x^2}{\sqrt{a^2-x^2}}\,dx =$$

$$= x\sqrt{a^2-x^2} - \int \left(\frac{-a^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right)dx =$$

$$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}\,dx =$$

$$= x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a} - I.$$

Уравнение отн. I: $I = x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a} - I$

Решим: $2I = x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}$

$$I = \boxed{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C}$$

табличный интеграл

Из № 1252, 1253 табл. инт-лы

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \sqrt{x^2+A}\,dx = \frac{x}{2}\sqrt{x^2+A} + \frac{A}{2}\ln\left|x+\sqrt{x^2+A}\right| + C$$

используем

№ 1228

$$\int x\arcsin x\,dx = \int \underbrace{\arcsin x}_{u}\cdot\underbrace{x\,dx}_{dv} =$$

$$= \int \underbrace{\arcsin x}_{u}\, \underbrace{d\left(\tfrac{1}{2}x^2\right)}_{dv} = \underbrace{\arcsin x}_{u}\cdot \underbrace{\tfrac{1}{2}x^2}_{v} - \int \underbrace{\tfrac{1}{2}x^2}_{v}\underbrace{d(\arcsin x)}_{du} =$$

$$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int x^2\frac{1}{\sqrt{1-x^2}}\,dx =$$

$$= \frac{1}{2}x^2\arcsin x + \frac{1}{2}\int \frac{-1+1-x^2}{\sqrt{1-x^2}}\,dx =$$

$$= \frac{1}{2}x^2\arcsin x + \frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right)dx =$$

$$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\boxed{\int\sqrt{1-x^2}\,dx} =$$

$$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\arcsin x + \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right) + C =$$

$$= \frac{1}{2}x^2\arcsin x - \frac{1}{4}\arcsin x + \frac{x}{4}\sqrt{1-x^2} + C$$