Дисциплина: Линейная алгебра и функции нескольких переменных

Семинар 11.

Частные производные функции нескольких переменных

При нахождении частной производной функции нескольких переменных по какой-либо переменной, остальные переменные полагаем постоянными.

I Найти частные производные 1-го и 2-го порядков.

№ 8.53 $\quad z = x^5 + y^5 - 5x^3y^3$

Решение.

$$\frac{\partial z}{\partial x} = 5x^4 - 15x^2y^3 \quad (y = const); \qquad \frac{\partial z}{\partial y} = 5y^4 - 15x^3y^2 \quad (x = const)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 20x^3 - 30xy^3$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 20y^3 - 30x^3y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -45x^2y^2; \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = -45x^2y^2$$

Отметим, что если функция $z = f(x, y)$ непрерывна в области D, то смешанные производные $2^{х}$ порядка равны, т.е. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

№ 8.59. $z = \frac{\cos y^2}{x}$

Решение.

$\underset{(y = const)}{\frac{\partial z}{\partial x}} = \cos y^2 \cdot \left(-\frac{1}{x^2}\right) = -\frac{\cos y^2}{x^2}$; $\underset{(x = const)}{\frac{\partial z}{\partial y}} = \frac{1}{x} \cdot (-\sin y^2) \cdot 2y = -\frac{2y \sin y^2}{x}$;

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\cos y^2}{x^2}\right) = -\cos y^2 \cdot \frac{-2}{x^3} = \frac{2\cos y^2}{x^3}$;

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{2y \sin y^2}{x}\right) = -\frac{2}{x}\left(\sin y^2 + y \cos y^2 \cdot 2y\right) = -\frac{2}{x}\left(\sin y^2 + 2y^2 \cos y^2\right)$;

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{2y}{x} \sin y^2\right) = \frac{2y \cdot \sin y^2}{x^2}$;

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{2y \sin y^2}{x^2}$.

№ 8.61. $z = \ln(x^2 + y^2)$

Решение. $\underset{(y = const)}{\frac{\partial z}{\partial x}} = \frac{1}{x^2 + y^2} \cdot 2x$; $\underset{(x = const)}{\frac{\partial z}{\partial y}} = \frac{2y}{x^2 + y^2}$;

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) =$$

$$= 2\cdot\frac{x^2+y^2-2x\cdot x}{(x^2+y^2)^2} = 2\,\frac{y^2-x^2}{(x^2+y^2)^2};$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) =$$

$$= 2\cdot\frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = 2\,\frac{x^2-y^2}{(x^2+y^2)^2};$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right) =$$

$$= 2y\cdot\left(-\frac{1}{(x^2+y^2)^2}\cdot 2x\right) = -\frac{4xy}{(x^2+y^2)^2}.$$

№ 8.64 $\quad u = \left(\frac{y}{x}\right)^z$

<u>Решение.</u> $\frac{\partial u}{\partial x} = z\cdot\left(\frac{y}{x}\right)^{z-1}\cdot y\cdot\left(-\frac{1}{x^2}\right) =$

$\begin{pmatrix} y = const \\ z = const \end{pmatrix}$ $\quad = -\frac{z\cdot y^z}{x^{z+1}} = -\frac{z}{x}\cdot\left(\frac{y}{x}\right)^z;$

$$\frac{\partial u}{\partial y} = z\left(\frac{y}{x}\right)^{z-1}\cdot\frac{1}{x} = \frac{z\cdot y^{z-1}}{x^z} = \frac{z}{y}\cdot\left(\frac{y}{x}\right)^z;$$

$\begin{pmatrix} x = const \\ z = const \end{pmatrix}$

$$\frac{\partial u}{\partial z} = \left(\frac{y}{x}\right)^z \ln\frac{y}{x};\begin{pmatrix} x = const \\ y = const \end{pmatrix}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{z}{x}\cdot\left(\frac{y}{x}\right)^z\right) = -\frac{zy^z\cdot(-(z+1))}{x^{z+2}} =$$

$\begin{pmatrix} y = const \\ z = const \end{pmatrix}$

$$= \frac{z(z+1)}{x^2}\cdot\left(\frac{y}{x}\right)^z;$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{z}{y}\cdot\left(\frac{y}{x}\right)^z\right) = \frac{\partial}{\partial y}\left(\frac{z\cdot y^{z-1}}{x^z}\right) =$$

$\left(\begin{array}{l} x = const \\ z = const \end{array}\right)$

$$= \frac{z}{x^z}\cdot(z-1)\,y^{z-2} = \frac{z\cdot(z-1)}{y^2}\cdot\left(\frac{y}{x}\right)^z;$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\left(\frac{y}{x}\right)^z\cdot \ln\frac{y}{x}\right) = \left(\frac{y}{x}\right)^z\cdot \ln^2\frac{y}{x};$$

$\left(\begin{array}{l} x = const \\ y = const \end{array}\right)$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{z\cdot y^{z-1}}{x^z}\right) = z\cdot y^{z-1}\cdot\left(-\frac{z}{x^{z+1}}\right) =$$

$$= -\frac{z^2}{xy}\cdot\left(\frac{y}{x}\right)^z;$$

$$\frac{\partial^2 u}{\partial x\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}\left(\left(\frac{y}{x}\right)^z \ln\frac{y}{x}\right) =$$

$$= y^z\cdot\left(-z\cdot\frac{1}{x^{z+1}}\cdot\ln\frac{y}{x} + \left(\frac{y}{x}\right)^z\cdot\frac{1}{\frac{y}{x}}\cdot y\left(-\frac{1}{x^2}\right)\right) =$$

$$= \left(\frac{y}{x}\right)^z\cdot\left(-\frac{z}{x}\ln\frac{y}{x} + \frac{x}{y}\cdot\left(-\frac{y}{x^2}\right)\right) =$$

$$= -\frac{1}{x}\cdot\left(\frac{y}{x}\right)^z\left(z\ln\frac{y}{x}+1\right);$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(\left(\frac{y}{x}\right)^z \ln\frac{y}{x}\right) = \frac{1}{x^z}\cdot z y^{z-1}\cdot\ln\frac{y}{x} +$$

$$+ \left(\frac{y}{x}\right)^z\cdot\frac{1}{\frac{y}{x}}\cdot\frac{1}{x} = \left(\frac{y}{x}\right)^z\cdot\frac{1}{y}\left(z\ln\frac{y}{x}+1\right).$$

$$\frac{\partial z}{\partial y} = \frac{1}{\cos^2 \frac{y^2}{x}} \cdot \frac{1}{x} \cdot 2y = \frac{2y}{x} \cdot \frac{1}{\cos^2 \frac{y^2}{x}};$$

Тогда $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ (формула)

$$dz = -\frac{y^2}{x^2} \cdot \frac{1}{\cos^2 \frac{y^2}{x}} dx + \frac{2y}{x} \cdot \frac{1}{\cos^2 \frac{y^2}{x}} dy.$$

№ 8.91. $z = \ln \cos \frac{x}{y}$

Решение. $\frac{\partial z}{\partial x} = \frac{1}{\cos \frac{x}{y}} \cdot \left(-\sin \frac{x}{y}\right) \cdot \frac{1}{y} =$

$$= -\frac{1}{y} \operatorname{tg} \frac{x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\cos \frac{x}{y}} \cdot \left(-\sin \frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y^2} \operatorname{tg} \frac{x}{y}$$

Тогда

$$dz = -\frac{1}{y} \operatorname{tg} \frac{x}{y} dx + \frac{x}{y^2} \operatorname{tg} \frac{x}{y} dy.$$

III Найти дифференциалы 1^го и 2^го порядков.

№ 8.103.

$$z = \sqrt{x^2 + 2xy}$$

Решение. $\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2 + 2xy}} \cdot (2x + 2y) =$

$$= \frac{x+y}{\sqrt{x^2+2xy}};$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+2xy}} \cdot 2x = \frac{x}{\sqrt{x^2+2xy}};$$

Тогда дифференциал 1^{го} порядка имеет вид

$$dz = \frac{x+y}{\sqrt{x^2+2xy}}\,dx + \frac{x}{\sqrt{x^2+2xy}}\,dy$$

Найдем $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x+y}{\sqrt{x^2+2xy}}\right) =$

$$= \frac{\sqrt{x^2+2xy} - \frac{1}{2\sqrt{x^2+2xy}}\cdot(2\cdot x+2y)\cdot(x+y)}{\left(\sqrt{x^2+2xy}\right)^2} =$$

$$= \frac{x^2+2xy-(x+y)^2}{(x^2+2xy)^{3/2}} = \frac{-y^2}{(x^2+2xy)^{3/2}};$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+2xy}}\right) = \frac{-x^2}{(x^2+2xy)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+2xy}}\right) =$$

$$= \frac{\sqrt{x^2+2xy} - \frac{1}{2\sqrt{x^2+2xy}}\cdot(2x+2y)\cdot x}{\left(\sqrt{x^2+2xy}\right)^2} =$$

$$= \frac{x^2+2xy-x^2-xy}{(x^2+2xy)^{3/2}} = \frac{xy}{(x^2+2xy)^{\frac{3}{2}}};$$

Тогда дифференциал 2^{го} рода имеет вид

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \text{ (формула)}$$

$$d^2z = \frac{-y^2}{(x^2+2xy)^{\frac{3}{2}}}dx^2 + 2\frac{xy}{(x^2+2xy)^{\frac{3}{2}}}dx\,dy + \frac{-x^2}{(x^2+2xy)^{\frac{3}{2}}}dy^2$$

№ 8.105 $\quad z=(x+y)e^{xy}$

Решение. $\dfrac{\partial z}{\partial x}=e^{xy}+(x+y)e^{xy}\cdot y=$

$=e^{xy}(1+xy+y^2)$;

$\dfrac{\partial z}{\partial y}=e^{xy}+(x+y)e^{xy}\cdot x=e^{xy}(1+x^2+xy)$;

Тогда дифференциал 1^го рода имеет вид

$dz=e^{xy}(y^2+xy+1)dx+e^{xy}(x^2+xy+1)dy$.

Найдем $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(e^{xy}(y^2+xy+1)\right)=$

$=e^{xy}\cdot y(y^2+xy+1)+e^{xy}\cdot y=e^{xy}y(y^2+xy+2)$;

$\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(e^{xy}(x^2+xy+1)\right)=$

$=e^{xy}\cdot x(x^2+xy+1)+e^{xy}\cdot x=e^{xy}x(x^2+xy+2)$.

Тогда дифференциал 2^го порядка имеет вид

$d^2z=e^{xy}y(y^2+xy+2)dx^2+2\dfrac{\partial^2 z}{\partial x\partial y}dxdy+$

$+e^{xy}x(x^2+xy+2)dy^2$

$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(e^{xy}(x^2+xy+1)\right)=e^{xy}y\cdot(x^2+xy+1)+$

$+e^{xy}(2x+y)=e^{xy}(x^2y+xy^2+2x+2y)$

В результате дифференциал $2^{го}$ порядка

$$d^2z = e^{xy}y(y^2+xy+2)\,dx^2 + 2e^{xy}(x^2y + xy^2 + 2x + 2y)\,dx\,dy + e^{xy}x(x^2+xy+2)\,dx^2.$$

Домашнее задание

Сборник задач по математике для ВТУЗов. Часть 2. Под ред. А.В. Ефимова и Б. П. Демидовича

№ 8,56, 8.58, 8.62, 8.92, 8.107