Дисциплина: Линейная алгебра и функции нескольких переменных

Семинар 12

Производная сложной функции нескольких переменных

I. Пусть функция

$u = f(x_1, x_2, ..., x_n)$ – дифференцируемая функция переменных $x_1, x_2, ..., x_n$, которые сами являются дифференцируемыми функциями

$x_1 = \varphi_1(t), x_2 = \varphi_2(t), ..., x_n = \varphi_n(t)$.

Тогда производная сложной функции $u = f(\varphi_1(t), \varphi_2(t), ..., \varphi_n(t))$ вычисляется по формуле

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + ... + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}.$$

II. Так, если функция $z = f(x, y)$ – дифференцируема по переменным x и y, а $x = \varphi(t)$, $y = \psi(t)$ – дифференцируемые функции, то производная функции $z = (\varphi(t), \psi(t))$ вычисляется по формуле

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial z}{\partial x}\cdot\frac{d\varphi}{dt} + \frac{\partial z}{\partial y}\cdot\frac{d\psi}{dt}.$$

III. Если $z = f(x,y)$, где $y = \varphi(x)$, то производная функции $z = f(x, \varphi(x))$ вычисляется по формуле

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{d\varphi}{dt}.$$

IV. Если $z = f(x,y)$, где $x = \varphi(\xi, \eta)$
$y = \psi(\xi, \eta)$,

то частные производные функции $z = f(\varphi(\xi,\eta), \psi(\xi,\eta))$ вычисляется по формулам

$$\frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \xi} = \frac{\partial z}{\partial x} \cdot \frac{\partial \varphi}{\partial \xi} + \frac{\partial z}{\partial y} \cdot \frac{\partial \psi}{\partial \xi}$$

$$\frac{\partial z}{\partial \eta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \eta} = \frac{\partial z}{\partial x} \cdot \frac{\partial \varphi}{\partial \eta} + \frac{\partial z}{\partial y} \cdot \frac{\partial \psi}{\partial \eta}$$

<u>№ 8.115</u>. $z = x^y$, $x = \ln t$, $y = \sin t$; $\frac{dz}{dt}$ – ?

<u>Решение</u>

$$\frac{\partial z}{\partial x} = y \cdot x^{y-1}, \quad \frac{\partial z}{\partial y} = x^y \cdot \ln x$$

$$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = \cos t$$

Тогда $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = yx^{y-1} \cdot \frac{1}{t} +$

$+ x^y \ln x \cdot \cos t \left(\begin{array}{l} \text{подставляем} \\ x = \ln t,\ y = \sin t \end{array} \right) =$

$$= \sin t \,(\ln t)^{\sin t - 1}\cdot \frac{1}{t} + (\ln t)^{\sin t}\cdot \cos t \cdot \ln(\ln t) =$$

$$= (\ln t)^{\sin t}\cdot\left(\frac{\sin t}{t\cdot \ln t} + \cos t \ln(\ln t)\right).$$

№ 8.119 $z = \operatorname{arctg}\dfrac{x+1}{y}$, где $y = e^{(x+1)^2}$, $\dfrac{dz}{dx}$ - ?

Решение. $\dfrac{\partial z}{\partial x} = \dfrac{1}{1+\left(\frac{x+1}{y}\right)^2}\cdot\dfrac{1}{y} = \dfrac{y}{y^2+(x+1)^2}$,

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x+1}{y}\right)^2}\cdot\left(-\frac{x+1}{y^2}\right) = -\frac{x+1}{y^2+(x+1)^2},$$

$$\frac{dy}{dx} = e^{(x+1)^2}\cdot 2(x+1).$$

Тогда $\dfrac{dz}{dx} = \dfrac{\partial z}{\partial x}\cdot\underbrace{\dfrac{dx}{dx}}_{=1} + \dfrac{\partial z}{\partial y}\cdot\dfrac{dy}{dx} =$

$$= \frac{y}{y^2+(x+1)^2} + \left(-\frac{x+1}{y^2+(x+1)^2}\right)\cdot e^{(x+1)^2}\cdot 2(x+1) =$$

$$= \frac{y - 2e^{(x+1)^2}(x+1)^2}{y^2+(x+1)^2} = \left\{\begin{array}{l}\text{заменим}\\ y = e^{(x+1)^2}\end{array}\right\} =$$

$$= \frac{e^{(x+1)^2}\left(1-2(x+1)^2\right)}{e^{(x+1)^2}+(x+1)^2}.$$

№ 8.122. $z = f(u, v)$, где $u = \dfrac{2y}{x+y}$, $v = x^2 - 3y$

$\dfrac{\partial z}{\partial x}$ - ? $\dfrac{\partial z}{\partial y}$ - ?

Решение.

$$\frac{\partial u}{\partial x} = -\frac{2y}{(x+y)^2}; \quad \frac{\partial u}{\partial y} = 2\,\frac{x+y-y}{(x+y)^2} = \frac{2x}{(x+y)^2};$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -3.$$

Тогда $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} =$

$$= \frac{\partial z}{\partial u}\cdot\left(-\frac{2y}{(x+y)^2}\right) + \frac{\partial z}{\partial v}\cdot 2x;$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} =$$

$$= \frac{\partial z}{\partial u}\cdot\frac{2x}{(x+y)^2} + \frac{\partial z}{\partial v}\cdot(-3).$$

№8.129. Показать, что функция $z = x\cdot f\left(\frac{y}{x}\right) - x^2 - y^2$ удовлетворяет уравнению $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - x^2 - y^2$. (1)

Решение. $\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + x\cdot\frac{\partial f\left(\frac{y}{x}\right)}{\partial x}\cdot\left(-\frac{y}{x^2}\right) - 2x$

$$\frac{\partial z}{\partial y} = x\cdot\frac{\partial f\left(\frac{y}{x}\right)}{\partial y}\cdot\frac{1}{x} - 2y.$$

Подставим $\frac{\partial z}{\partial x}$ и $\frac{\partial z}{\partial y}$ в выражение (1)

$$x\cdot\left(f\left(\frac{y}{x}\right) - \frac{\partial f\left(\frac{y}{x}\right)}{\partial x}\cdot\frac{y}{x} - 2x\right) + y\left(x\,\frac{\partial f\left(\frac{y}{x}\right)}{\partial y}\cdot\frac{1}{x} - 2y\right) =$$

$$= z - x^2 - y^2;$$

$$x\cdot f\left(\frac{y}{x}\right)-y\frac{\partial f\left(\frac{y}{x}\right)}{\partial x}-2x^2+y\frac{\partial f\left(\frac{y}{x}\right)}{\partial y}-2y^2=$$
$$=z-x^2-y^2;$$
$$x f\left(\frac{y}{x}\right)-2x^2-2y^2=z-x^2-y^2$$
$$\underbrace{x f\left(\frac{y}{x}\right)-x^2-y^2}_{=z}-x^2-y^2=z-x^2-y^2$$

верно.

№ 8.120 $z=u^2\cdot\ln v$, где $u=\frac{y}{x}$, $v=x^2+y^2$

$\frac{\partial z}{\partial x}$ -? , $\frac{\partial z}{\partial y}$ -?

Решение. $\frac{\partial z}{\partial u}=2u\cdot\ln v$; $\frac{\partial z}{\partial v}=\frac{u^2}{v}$;

$\frac{\partial u}{\partial x}=-\frac{y}{x^2}$; $\frac{\partial u}{\partial y}=\frac{1}{x}$; $\frac{\partial v}{\partial x}=2x$; $\frac{\partial v}{\partial y}=2y$

Тогда
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=$$
$$=2u\cdot\ln v\cdot\left(-\frac{y}{x^2}\right)+\frac{u^2}{v}\cdot 2x=$$
$$=2\frac{y}{x}\cdot\ln(x^2+y^2)\cdot\left(-\frac{y}{x}\right)+\frac{y^2}{x^2(x^2+y^2)}\cdot 2y=$$
$$=\frac{y^2}{x^2}\left(-2\ln(x^2+y^2)+\frac{2y}{x^2+y^2}\right);$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2u\ln v\cdot\frac{1}{x}+\frac{u^2}{v}\cdot 2y=$$
$$=2\frac{y}{x}\cdot\ln(x^2+y^2)\cdot\frac{1}{x}+\frac{y^2}{x^2(x^2+y^2)}\cdot 2y=2\frac{y}{x}\left(\frac{\ln(x^2+y^2)}{x}+\frac{y^2}{x(x^2+y^2)}\right).$$

Производная неявной функции нескольких переменных

Пусть функция задана неявно уравнением $F(x,y)=0$, $y=f(x)$. Тогда

$$\frac{dy}{dx} = -\frac{F'_x(x,y)}{F'_y(x,y)}.$$

№ 8.141. $y\cdot\sin x-\cos(x-y)=0$, $\frac{dy}{dx}$ – ?

Решение. $F(x,y)=y\cdot\sin x-\cos(x-y)$

$$F'_x(x,y)=\frac{\partial F(x,y)}{\partial x}=y\cos x+\sin(x-y)$$

$$F'_y(x,y)=\sin x+\sin(x-y)\cdot(-1)=\frac{\partial F(x,y)}{\partial y}$$

Тогда $\frac{dy}{dx}=-\frac{y\cos x+\sin(x-y)}{\sin x-\sin(x-y)}$.

№ 8.146. $z\cdot\ln(x+z)-\frac{xy}{z}=0$, $\frac{\partial z}{\partial x}$ – ? ; $\frac{\partial z}{\partial y}$ – ?

Решение. $F(x,y,z)=z\cdot\ln(x+z)-\frac{xy}{z}$

$$F'_x(x,y,z)=z\cdot\frac{1}{x+z}-\frac{y}{z}=\frac{z^2-y(x+z)}{(x+z)\cdot z};$$

$$F'_y(x,y,z)=-\frac{x}{z};$$

$$F'_z=\ln(x+z)+\frac{z}{x+z}+\frac{xy}{z^2}=\left(\ln(x+z)=\frac{xy}{z^2} \text{ из исходной функции}\right)=$$

$$=\frac{xy}{z^2}+\frac{z}{x+z}+\frac{xy}{z^2}=\frac{2xy(x+y)+z^3}{(x+z)\cdot z^2};$$

$$\frac{\partial z}{\partial x} = -\frac{F'_x(x,y,z)}{F'_z(x,y,z)} = -\frac{\dfrac{z^2 - y(x+z)}{(x+z)\cdot z}}{\dfrac{2xy(x+y)+z^3}{(x+z)\cdot z^2}} =$$

$$= -\frac{z(z^2 - y(x+z))}{2xy(x+y)+z^3} = \frac{yz(x+z) - z^3}{2xy(x+y)+z^3};$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y(x,y,z)}{F'_z(x,y,z)} = -\frac{-\dfrac{x}{z}}{\dfrac{2xy(x+y)+z^3}{(x+z)\cdot z^2}} =$$

$$= \frac{xz(x+z)}{2xy(x+y)+z^3}.$$

№ 8.152 $\quad x + y^2 + z = e^z, \quad \frac{\partial^2 z}{\partial x^2} - ?\ \frac{\partial^2 z}{\partial y^2} - ?\ \frac{\partial^2 z}{\partial x \partial y} - ?$

Решение. $\quad F(x,y,z) = x + y^2 + z - e^z.$

$$F'_x = \frac{\partial F(x,y,z)}{\partial x} = 1; \quad F'_y(x,y,z) = \frac{\partial F}{\partial y} = 2y;$$

$$F'_z(x,y,z) = \frac{\partial F(x,y,z)}{\partial z} = 1 - e^z.$$

Тогда $\frac{\partial z}{\partial x} = -\frac{F'_x(x,y,z)}{F'_z(x,y,z)} = -\frac{1}{1-e^z} = \frac{1}{e^z - 1} =$

$$= \left\{ \begin{matrix} \text{из исходной функции} \\ e^z = x + y^2 + z \end{matrix} \right\} = \frac{1}{x + y + z - 1}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{e^z - 1}\right) = -\frac{e^z \cdot \frac{\partial z}{\partial x}}{(e^z - 1)^2} =$$

$$= -\frac{(x+y^2+z)\cdot \frac{1}{x+y^2+z-1}}{(x+y^2+z-1)^2} = -\frac{x+y^2+z}{(x+y^2+z-1)^3};$$

$$\frac{\partial z}{\partial y} = -\frac{F'_y(x,y,z)}{F'_z(x,y,z)} = -\frac{2y}{1-e^z} = \frac{2y}{e^z-1} =$$

$$= \frac{2y}{x+y+z-1};$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{e^z-1}\right) =$$

$$= 2\,\frac{e^z-1-ye^z\frac{\partial z}{\partial y}}{(e^z-1)^2} = 2\,\frac{x+y^2+z-1-y\frac{(x+y^2+z)}{x+y^2+z-1}\cdot\frac{2y}{6}}{(x+y^2+z-1)^2} =$$

$$= 2\,\frac{(x+y^2+z-1)^2-2y^2(x+y^2+z)}{(x+y^2+z-1)^3};$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{e^z-1}\right) = 2y\,\frac{e^z\frac{\partial z}{\partial x}}{(e^z-1)^2} =$$

$$= \frac{2y(x+y^2+z)\cdot\frac{1}{x+y^2+z-1}}{(x+y^2+z-1)^2} = \frac{2y(x+y^2+z)}{(x+y^2+z-1)^3}.$$

Домашнее задание.

№ 8.116, 8.118, 8.124, 8.128, 8.140, 8.149, 8.151.