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Surface and interfacial waves of arbitrary form in isotropic elastic media

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ABSTRACT

A method due to Friedlander of accommodating disturbances of arbitrary form into the theory of surface waves in a semi-infinite isotropic elastic body is extended and shown to yield a simple closed form solution for the displacement field. An analogous treatment of interfacial waves of arbitrary form at a plane contact discontinuity separating different isotropic elastic materials is also given.

RÉSUMÉ

On développe une méthode, conçue par Friedlander, qui fait entrer les perturbations de forme arbitraire dans la théorie des ondes de surface dans un corps élastique isotropique semi-infini, et on montre qu'elle permet d'obtenir une solution simple et exacte pour le champ de déplacement. Les ondes de forme arbitraire qui existent dans le plan à la frontière de materiaux élastiques isotropiques differents sont traitées de façon analogue.

1. Introduction

In conformity with Lord Rayleigh's original work [1] textbook accounts of free surface waves in an isotropic elastic half-space with traction-free boundary are invariably developed in terms of sinusoidal waveforms (see, for example, [2] to [8]). But Rayleigh waves are non-dispersive, as are Stoneley waves associated with an interface between dissimilar isotropic elastic half-spaces [9], and the possibility of a theory of surface and interfacial waves of arbitrary form therefore suggests itself. In fact the essential ingredients of such an analysis were provided as long ago as 1948 by Friedlander in a paper [10] which appears to have been widely overlooked by subsequent writers on classical elastodynamics. Focussing attention on the surface disturbance resulting from a Rayleigh-type wave of general form, Friedlander reached the elegant conclusion that the boundary values of the two non-zero displacement components are, in essence, Hilbert transforms of one another.[1] He stopped short, however, of a complete representation of the wave field.

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[1] See equations (2.18) below.

In this paper certain extensions to Friedlander's ideas are made, centering on the use of conjugate harmonic functions, and a full solution of the surface wave problem is obtained. A parallel treatment of free interfacial waves of arbitrary shape is also presented.

2. Surface waves

(a) *Basic equations.* Let x_1, x_2, x_3 be rectangular Cartesian coordinates and suppose that the semi-infinite region $x_3 \geqq 0$ is a natural configuration of a homogeneous body composed of isotropic elastic material. In the absence of body forces a small amplitude disturbance of this configuration is governed by the equations of motion

$$(c_1^2 - c_2^2)u_{p,ip} + c_2^2 u_{i,pp} = \partial^2 u_i/\partial t^2 \tag{2.1}$$

in which u_i $(i = 1, 2, 3)$ are the components of displacement, c_1 and c_2 are respectively the constant speeds of propagation of dilatational and rotational body waves, t is the time and standard conventions of indicial notation are adopted.

An in-plane motion of the half-space is one in which the displacement is universally parallel to a fixed plane through the x_3-axis and, at all times, is everywhere the same on lines orthogonal to this plane. Without loss of generality the reference plane can be taken to be $x_2 = 0$. Then, as may easily be verified, if $\phi(x_1, x_3, t)$ and $\psi(x_1, x_3, t)$ are scalar functions satisfying the wave equations

$$c_1^2(\phi_{,11} + \phi_{,33}) = \partial^2\phi/\partial t^2, \qquad c_2^2(\psi_{,11} + \psi_{,33}) = \partial^2\psi/\partial t^2, \tag{2.2}$$

a solution of equations (2.1) representing an in-plane motion is given by

$$u_1 = \phi_{,1} - \psi_{,3}, \qquad u_2 = 0, \qquad u_3 = \phi_{,3} + \psi_{,1}. \tag{2.3}$$

The assumption that the boundary of the half-space is traction-free gives rise to the conditions

$$\left.\begin{aligned} 2\phi_{,13} + \psi_{,11} - \psi_{,33} &= 0 \\ (1-2\Lambda)\phi_{,11} + \phi_{,33} + 2\Lambda\psi_{,13} &= 0 \end{aligned}\right\} \text{ at } x_3 = 0 \text{ for all } x_1 \text{ and } t, \tag{2.4}$$

where

$$\Lambda = c_2^2/c_1^2 \; (0 < \Lambda < 1).$$

(b) *Representation of a surface wave by two plane harmonic functions.* Friedlander's discussion of elastic surface waves of arbitrary form stems from the following observation. Let $f(x, y)$ and $g(x, y)$ be functions which are harmonic in the half plane $y > 0$, continuously differentiable on the line $y = 0$ and which, together with their first partial derivatives, tend to zero as $(x^2 + y^2)^{\frac{1}{2}} \to \infty$ for all $y > 0$. Further, let v be a positive constant, with the physical dimensions of velocity, such that

$$0 < \gamma < 1 \text{ where } \gamma = v^2/c_2^2.$$

Then the expressions

$$\left.\begin{aligned} \phi(x_1, x_3, t) &= f(x_1 - vt, (1-\Lambda\gamma)^{\frac{1}{2}}x_3), \\ \psi(x_1, x_3, t) &= g(x_1 - vt, (1-\gamma)^{\frac{1}{2}}x_3), \end{aligned}\right\} \tag{2.5}$$

solve the wave equations (2.2) in the region $x_3 > 0$ for all x_1 and t. The associated non-zero displacement components, derived from (2.3), are

$$\left.\begin{aligned} u_1(x_1, x_3, t) &= f_x(x_1 - vt, (1-\Lambda\gamma)^{\frac{1}{2}}x_3) - (1-\gamma)^{\frac{1}{2}} g_y(x_1 - vt, (1-\gamma)^{\frac{1}{2}}x_3), \\ u_3(x_1, x_3, t) &= (1-\Lambda\gamma)^{\frac{1}{2}} f_y(x_1 - vt, (1-\Lambda\gamma)^{\frac{1}{2}}x_3) + g_x(x_1 - vt, (1-\gamma)^{\frac{1}{2}}x_3), \end{aligned}\right\} \tag{2.6}$$

a suffix notation being employed for the partial derivatives of f and g.

Equations (2.6) clearly describe a progressive wave, travelling with constant speed v and without change of shape in the positive x_1-direction, and with the accompanying displacement decaying towards zero as $x_3 \to \infty$. The in-plane motion generated by any pair of plane harmonic functions with the above-mentioned properties therefore has the behaviour customarily taken to characterize a free surface wave. It is not permissible, however, to assign the two functions f and g in a completely arbitrary manner since the boundary conditions (2.4) remain to be satisfied. Applied to the wave functions (2.5) they supply the relations

$$\left.\begin{aligned} 2(1-\Lambda\gamma)^{\frac{1}{2}} f_y(x, 0) + (2-\gamma) g_x(x, 0) &= 0 \\ -(2-\gamma) f_x(x, 0) + 2(1-\gamma)^{\frac{1}{2}} g_y(x, 0) &= 0 \end{aligned}\right\} \quad \text{for all } x, \tag{2.7}$$

an integration with respect to x having been carried out.[2]

(c) *Expression of the displacement field in terms of a single function.* The starting point described in the last subsection will now be slightly modified. Let $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ be real-valued functions which are continuously differentiable for all x and are such that the complex-valued infinite integrals

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\mathfrak{f}(\xi)\,d\xi}{\xi - z} \quad \text{and} \quad \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\mathfrak{g}(\xi)\,d\xi}{\xi - z} \qquad (z = x + iy) \tag{2.8}$$

exist for all $y \geqq 0$ and tend to zero as $|z| \to \infty$ for all $y > 0$.[3] Then the real parts of the integrals (2.8),

$$f(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\mathfrak{f}(\xi)\,d\xi}{(\xi - x)^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\mathfrak{g}(\xi)\,d\xi}{(\xi - x)^2 + y^2}, \tag{2.9}$$

are plane harmonic functions with the properties in the half-plane $y \geqq 0$ stipulated in Section 2(b). The imaginary parts, $f^*(x, y)$ and $g^*(x, y)$, also have these properties and are harmonic conjugates of $f(x, y)$ and $g(x, y)$ respectively. The boundary values of the conjugate pairs (f, f^*) and (g, g^*) on the line $y = 0$ are mutually reciprocal under the Hilbert transform. Thus

$$f(x, 0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f^*(\xi, 0)}{\xi - x}\,d\xi, \qquad f^*(x, 0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi, 0)}{\xi - x}\,d\xi, \tag{2.10}$$

---

[2] The constants of integration must be equated to zero since otherwise there is an eventual contribution to the displacement field which does not conform to the required behaviour at infinity.

[3] Sufficient conditions are that $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are Hölder continuous for all finite x and $O(|x|^{-\alpha})$ $(\alpha > 0)$ as $x \to \pm\infty$ (see [11, pp. 109–112]). These conditions are not necessary, however; witness the example (2.17) given later.

with similar relations for $g(x, 0)$ and $g^*(x, 0)$, P signifying a Cauchy principal value. It will transpire that a free surface wave can be constructed from the single function $\mathfrak{f}$, restricted only by the analytical requirements indicated above.

With the aid of the second and fourth of the Cauchy-Riemann equations,

$$f_x = f_y^*, \quad f_y = -f_x^*; \qquad g_x = g_y^*, \quad g_y = -g_x^*, \tag{2.11}$$

the boundary conditions (2.7) can be expressed entirely in terms of partial derivatives with respect to x. A second integration can then be performed to yield

$$d_1(x, 0) = 0, \quad d_2(x, 0) = 0 \quad \text{for all } x, \tag{2.12}$$

where

$$\left.\begin{aligned} d_1(x, y) &= -2(1-\Lambda\gamma)^{\frac{1}{2}}f^*(x, y)+(2-\gamma)g(x, y), \\ d_2(x, y) &= (2-\gamma)f(x, y)+2(1-\gamma)^{\frac{1}{2}}g^*(x, y). \end{aligned}\right\} \tag{2.13}$$

The functions d_1 and d_2 are harmonic in the half-plane $y > 0$ and, on account of the conditions imposed on $\mathfrak{f}$ and $\mathfrak{g}$, bounded there. They can therefore be represented in terms of their boundary values on the line $y = 0$ as Poisson integrals [12, p. 169]. According to (2.12), however, the boundary data are everywhere zero. Hence d_1 and d_2 vanish identically for $y \geqq 0$ and equations (2.13) provide the connexions

$$f^*(x, y) = \frac{2-\gamma}{2(1-\Lambda\gamma)^{\frac{1}{2}}}\, g(x, y), \qquad g^*(x, y) = -\frac{2-\gamma}{2(1-\gamma)^{\frac{1}{2}}}\, f(x, y). \tag{2.14}$$

Since the harmonic conjugate of f^* is $-f$ (i.e. $f^{**} = -f$), the relation

$$(2-\gamma)^2-4(1-\gamma)^{\frac{1}{2}}(1-\Lambda\gamma)^{\frac{1}{2}} = 0 \tag{2.15}$$

is seen to be necessary and sufficient for the consistency of equations (2.14). This condition, which is recognized as the secular equation determining the speed of propagation of Rayleigh waves in an isotropic elastic half-space, is discussed further in Section 2(e).

Equations (2.14) are alternative specifications of the relationship between the two plane harmonic functions f and g, and the explicit formula

$$g(x, y) = \frac{4(1-\Lambda\gamma)^{\frac{1}{2}}}{2-\gamma} \operatorname{Im} f(\tfrac{1}{2}(x+iy), \tfrac{1}{2}(y-ix))$$

may be noted in passing (cf. [12, p. 27]). It is now possible to eliminate g from the displacement expressions (2.6) by the use of $(2.14)_2$ in conjunction with the Cauchy-Riemann equations $(2.11)_{3,4}$: the final results obtained are

$$\left.\begin{aligned} u_1(x_1, x_3, t) &= f_x(x_1-vt, (1-\Lambda\gamma)^{\frac{1}{2}}x_3)-\tfrac{1}{2}(2-\gamma)f_x(x_1-vt, (1-\gamma)^{\frac{1}{2}}x^3), \\ u_3(x_1, x_3, t) &= \frac{2(1-\Lambda\gamma)^{\frac{1}{2}}}{2-\gamma}\{\tfrac{1}{2}(2-\gamma)f_y(x_1-vt,\ (1-\Lambda\gamma)^{\frac{1}{2}}x_3) \\ &\qquad\qquad -f_y(x_1-vt,\ (1-\gamma)^{\frac{1}{2}}x_3)\}, \end{aligned}\right\} \tag{2.16}$$

$(2.16)_2$ having been rearranged with the aid of (2.15).

When the real-valued function $\mathfrak{f}$ is prescribed the plane harmonic function f is given by $(2.9)_1$ and equations (2.16) then furnish the non-zero displacement components associated with a progressive free surface wave travelling in the x_1-direction with the Rayleigh wave speed. It is left to the reader to verify that the standard results for a sinusoidal Rayleigh wave of angular frequency ω are recovered on setting

$$\mathfrak{f}(x) = (av/\omega)\cos\{(\omega/v)x\}, \tag{2.17}$$

where a is a constant.

(d) *The boundary displacement.* The components of displacement at the boundary $x_3 = 0$ are given by equations (2.16) as

$$u_1(x_1, 0, t) = \tfrac{1}{2}\gamma f_x(x_1 - vt, 0) \equiv \bar{u}_1(x_1 - vt),$$

$$u_3(x_1, 0, t) = -\frac{\gamma(1-\Lambda\gamma)^{\frac{1}{2}}}{2-\gamma} f_y(x_1 - vt, 0) \equiv \bar{u}_3(x_1 - vt).$$

Since f_x and f_y are plane harmonic functions, with f_x the conjugate of f_y, there follow directly from equations (2.10) the relations

$$\left.\begin{aligned} \bar{u}_1(x_1 - vt) &= \frac{2-\gamma}{2\pi(1-\Lambda\gamma)^{\frac{1}{2}}} P\int_{-\infty}^{\infty} \frac{\bar{u}_3(\xi)\,d\xi}{\xi - x_1 + vt}, \\ \bar{u}_3(x_1 - vt) &= -\frac{2(1-\Lambda\gamma)^{\frac{1}{2}}}{\pi(2-\gamma)} P\int_{-\infty}^{\infty} \frac{\bar{u}_1(\xi)\,d\xi}{\xi - x_1 + vt}, \end{aligned}\right\} \tag{2.18}$$

first obtained by Friedlander [10, p. 383].

(e) *Existence and uniqueness of the surface wave.* The procedure described in Section 2(c) yields a unique surface wave if and only if the secular equation (2.15) determines precisely one real value of γ between 0 and 1.[4] A simple direct proof of this property, subject to the inequalities $0 < \Lambda < 1$,[5] can be based upon the function

$$\Gamma(\gamma) = (2-\gamma)^2(1-\gamma)^{-\frac{1}{2}}(1-\Lambda\gamma)^{-\frac{1}{2}} - 4$$

which clearly has the same real zeros in the interval (0, 1) as the left side of equation (2.15). Since

$$\Gamma(\gamma) = \begin{cases} -2(1-\Lambda)\gamma + O(\gamma^2) \text{ as } \gamma \to 0, \\ (1-\Lambda)^{-\frac{1}{2}}(1-\gamma)^{-\frac{1}{2}} + O(1) \text{ as } \gamma \to 1, \end{cases}$$

$\Gamma(\gamma)$ is negative near $\gamma = 0$ and positive near $\gamma = 1$. And because

$$\Gamma''(\gamma) = \frac{\{2(1-2\Lambda+3\Lambda^2)-(1-4\Lambda+7\Lambda^2)\gamma\}^2 + 2(1-\Lambda)^2(1-2\Lambda)^2\gamma^2}{4\{(1-\Lambda)^2+2\Lambda^2\}(1-\gamma)^{\frac{5}{2}}(1-\Lambda\gamma)^{\frac{5}{2}}}$$

is positive for all $0 < \gamma < 1$, Γ has only one zero in this interval.

---

[4] The possibility of (2.15) having complex roots is not relevant to the present discussion, but it seems not to be generally realized that the association of a sinusoidal Rayleigh wave with a complex root of the secular equation is ruled out *a priori* by Burridge's proof [13, §6] that no such wave can be damped.

[5] The positive definiteness of the elastic strain-energy function implies that $0 < \Lambda < \frac{3}{4}$.

3. Interfacial waves

In this section the transmitting medium is assumed to consist of two different homogeneous bodies each possessing a natural configuration in which the constituent material occupies a semi-infinite region and is isotropic. The two bodies are taken to be in welded contact and the undisturbed surface of separation is identified with the coordinate plane $x_3 = 0$. The extension of the theory developed in Section 2 to interfacial waves in this composite body is quite straightforward and only the essential details of the analysis will be given. The notation used previously is applied to the material occupying the region $x_3 > 0$ and the same symbols with primes attached refer to the corresponding quantities for the other constituent. The densities and shear moduli of the two materials are denoted by ρ, ρ' and μ, μ' respectively.

(a) *The displacement field.* Let $f(x, y)$, $g(x, y)$, $f'(x, y)$ and $g'(x, y)$ be plane harmonic functions, each derived from a suitable real-valued function in the manner of equations (2.9). Then the wave functions

$$\left.\begin{aligned}\phi(x_1, x_3, t) &= f(x_1 - vt,\ p_1 x_3)\\ \psi(x_1, x_3, t) &= g(x_1 - vt,\ p_2 x_3)\end{aligned}\right\} x_3 > 0, \qquad \left.\begin{aligned}\phi'(x_1, x_3, t) &= f'(x_1 - vt,\ -p'_1 x_3)\\ \psi'(x_1, x_3, t) &= g'(x_1 - vt,\ -p'_2 x_3)\end{aligned}\right\} x_3 < 0,$$

in which

$$p_\alpha = \{1 - (v/c_\alpha)^2\}^{\frac{1}{2}}, \qquad p'_\alpha = \{1 - (v/c'_\alpha)^2\}^{\frac{1}{2}} \qquad (\alpha = 1, 2),$$

satisfy equations (2.2) and their counterparts valid in the region $x_3 < 0$. The associated non-zero displacement components,

$$\left.\begin{aligned}&\left.\begin{aligned}u_1(x_1, x_3, t) &= f_x(x_1 - vt,\ p_1 x_3) - p_2 g_y(x_1 - vt,\ p_2 x_3)\\ u_3(x_1, x_3, t) &= p_1 f_y(x_1 - vt,\ p_1 x_3) + g_x(x_1 - vt,\ p_2 x_3)\end{aligned}\right\} x_3 > 0,\\ &\left.\begin{aligned}u'_1(x_1, x_3, t) &= f'_x(x_1 - vt,\ -p'_1 x_3) + p'_2 g'_y(x_1 - vt,\ -p'_2 x_3)\\ u'_3(x_1, x_3, t) &= -p'_1 f'_y(x_1 - vt,\ -p'_1 x_3) + g'_x(x_1 - vt,\ -p'_2 x_3)\end{aligned}\right\} x_3 < 0,\end{aligned}\right\} \tag{3.1}$$

evidently describe a progressive in-plane wave, travelling in the x_1-direction with constant speed v and falling off in intensity with increasing distance from the junction $x_3 = 0$. In short, equations (3.1) represent a free interfacial wave of arbitrary form.

(b) *The secular equation.* At the interface $x_3 = 0$ the displacement and stress vectors are continuous. These conditions entail four linear relations among the partial derivatives of f, g, f' and g', two involving first derivatives and the others second derivatives. The introduction of the conjugate harmonic functions f^*, g^*, f'^* and g'^* makes possible the integration of these relations and the results, corresponding to (2.12), are

$$e_A(x, 0) = 0 \ (A = 1, 2, 3, 4) \text{ for all } x,$$

where

$$e_1(x, y) = f(x, y) - f'(x, y) + p_2 g^*(x, y) + p'_2 g'^*(x, y),$$

$$
\begin{aligned}
e_2(x,y) &= -p_1 f^*(x,y) - p_1' f'^*(x,y) + g(x,y) - g'(x,y),\\
e_3(x,y) &= -2\mu p_1 f^*(x,y) - 2\mu' p_1' f'^*(x,y) + \mu(1+p_2^2)g(x,y) - \mu'(1+p_2'^2)g'(x,y),\\
e_4(x,y) &= -\mu(1+p_2^2)f(x,y) + \mu'(1+p_2'^2)f'(x,y) - 2\mu p_2 g^*(x,y) - 2\mu' p_2' g'^*(x,y).
\end{aligned}
$$

The considerations, relating to the Dirichlet problem for the half-plane, which were used in Section 2(c) to secure equations (2.14) now imply that each of the functions e_A is identically zero for $y \geqq 0$. The resulting connexions between f, g, f', g' and their harmonic conjugates can be put into the matrix form

$$\boldsymbol{f}^*(x,y) = \boldsymbol{C}_1 \boldsymbol{g}(x,y), \quad \boldsymbol{g}^*(x,y) = -\boldsymbol{C}_2 \boldsymbol{f}(x,y), \tag{3.2}$$

where $\boldsymbol{f} = (f, f')^T$ etc. and[6]

$$\boldsymbol{C}_\alpha = \begin{pmatrix} (\kappa - \rho v^2)/\kappa p_\alpha & \rho' v^2/\kappa p_\alpha \\ \rho v^2/\kappa p_\alpha' & -(\kappa + \rho' v^2)/\kappa p_\alpha' \end{pmatrix} \quad (\alpha = 1, 2) \tag{3.3}$$

with $\kappa = 2(\mu - \mu')$. Since $f^{**} = -f$ etc., it follows from equations (3.2) that

$$\boldsymbol{g}(x,y) = \boldsymbol{C}_2 \boldsymbol{f}^*(x,y) \text{ and } (\boldsymbol{I} - \boldsymbol{C}_1\boldsymbol{C}_2)\boldsymbol{f}(x,y) = 0, \tag{3.4}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix, and from $(3.4)_2$ the relation

$$\det(\boldsymbol{I} - \boldsymbol{C}_1\boldsymbol{C}_2) = 0 \tag{3.5}$$

is seen to be a necessary and sufficient condition for the consistency of equations (3.2). The secular equation

$$
\begin{aligned}
&v^4\{(\rho-\rho')^2 - (\rho p_1' + \rho' p_1)(\rho p_2' + \rho' p_2)\}\\
&\quad + 2\kappa v^2\{\rho(p_1' p_2' - 1) - \rho'(p_1 p_2 - 1)\} + \kappa^2(p_1 p_2 - 1)(p_1' p_2' - 1) = 0,
\end{aligned} \tag{3.6}
$$

first derived by Stoneley [9, p. 419] for sinusoidal waves, is now obtained on combining equations (3.3) and (3.5) and expanding the determinant.

(c) *Completion of the solution.* Equation $(3.4)_1$ relates the functions g and g' to f^* and f'^*, and from $(3.4)_2$ the quotient f'/f can be determined. Thus, with the aid of the appropriate Cauchy-Riemann equations, the displacement equations (3.1) can be expressed entirely in terms of the single plane harmonic function f. The end-results of this calculation are

$$
\left.
\begin{aligned}
&\left.\begin{aligned}
u_1(x_1,x_3,t) &= f_x(x_1 - vt, p_1 x_3) - n p_2 f_x(x_1 - vt, p_2 x_3)\\
u_3(x_1,x_3,t) &= p_1 f_y(x_1 - vt, p_1 x_3) - n f_y(x_1 - vt, p_2 x_3)
\end{aligned}\right\} x_3 \geqq 0,\\
&\left.\begin{aligned}
u_1'(x_1,x_3,t) &= m f_x(x_1 - vt, -p_1' x_3) + m n' p_2' f_x(x_1 - vt, -p_2' x_3)\\
u_3'(x_1,x_3,t) &= -m p_1' f_y(x_1 - vt, -p_1' x_3) - m n' f_y(x_1 - vt, -p_2' x_3)
\end{aligned}\right\} x_3 \leqq 0,
\end{aligned}
\right\} \tag{3.7}
$$

where

---

[6] Note that $c_2^2 = \mu/\rho$, $c_2'^2 = \mu'/\rho'$.

$$n = \frac{\kappa(p_1 p_2' - 1) + (\rho - \rho')v^2}{\kappa(p_2' - p_2) - (\rho p_2' + \rho' p_2)v^2}, \quad n' = \frac{\kappa(p_1' p_2 - 1) + (\rho - \rho')v^2}{\kappa(p_2' - p_2) - (\rho p_2' + \rho' p_2)v^2},$$

and

$$m = \frac{1 - np_2}{1 + n'p_2'} = \frac{n - p_1}{n' + p_1'}. \tag{3.8}$$

The non-zero components of displacement at the surface of separation of the constituent materials are seen from equations (3.7) and (3.8) to be given by

$$u_1(x_1, 0, t) = u_1'(x_1, 0, t) = (1 - np_2) f_x(x_1 - vt, 0),$$

$$u_3(x_1, 0, t) = u_3'(x_1, 0, t) = -(n - p_1) f_y(x_1 - vt, 0).$$

Each of these components can be expressed as a constant multiple of the Hilbert transform of the other, exactly as Section 2(d).

(d) *Existence-uniqueness considerations.* Equations (3.7) represent a unique interfacial wave if and only if the secular equation (3.6) determines exactly one positive real value of v^2 less than the smaller of c_2^2 and $c_2'^2$. This question has been investigated by Scholte [14] and is not pursued further here.

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