

Explicit models for elastic and piezoelectric surface waves

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Explicit models for the Rayleigh and Bleustein–Gulyaev surface waves are extracted from the original 2D formulations within the general framework of linear elasticity and electroelasticity. The derivations are based on perturbing in slow time the self-similar solutions for homogeneous surface wave. Both the proposed models involve hyperbolic equations on the surface along with elliptic equations over the interior, emphasizing the dual nature of a surface wave. Comparisons with exact solutions are presented, including that for the plane Lamb problem.

Keywords: elastic; piezoelectric; surface wave; Rayleigh wave; Bleustein–Gulyaev wave.

1. Introduction

Surface waves seem to be hidden, in a sense, in underlying mathematical formulations. In particular, the speed of the famous Rayleigh wave (Lord Rayleigh, 1885) is not an explicit parameter in the equations of linear elasticity. In order to extract it, one usually has to proceed to a transcendental dispersion relation (see (3.15)). The evident importance of the Rayleigh wave motivates an alternative analysis under more general assumptions.

An important initial insight into the propagation of Rayleigh waves was made by Friedlander (1948), who presented a self-similar solution of the homogeneous problem for an elastic half-plane in terms of arbitrary plane harmonic functions. As might be expected, the associated wave speed coincided with the Rayleigh speed. In a later publication, Chadwick (1976) demonstrated that the solution in Friedlander (1948) may be expressed in terms of a single plane harmonic function. In the above-mentioned paper, Chadwick also extended his methodology to the interfacial Stoneley wave. Among the publications on the subject, we also mention the paper by Knowles (1966) dealing with some generalizations for the Rayleigh waves sinusoidal in time.

Recent attempts related to the construction of approximate theories for nondispersive elastic and electroelastic surface waves induced by impact concentrated loads were reported briefly in Kaplunov & Kossovich (2004) and Kaplunov *et al.* (2004). Both these notes utilize the so-called symbolic operator method proposed by Lur'e (1964). According to this method partial derivatives along the surface are treated as symbolic operators, which makes it possible to reduce formally the original partial differential equations to ordinary differential equations. See Kaplunov *et al.* (1998) for more detail.

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In this paper, we reconsider the problems formulated in Kaplunov & Kossovich (2004) and Kaplunov *et al.* (2004). To satisfy the inhomogeneous boundary conditions on the surface, we perturb in slow time the self-similar solutions given in Friedlander (1948) and Chadwick (1976). The main result of the paper is the derivation of explicit models for surface waves only. The Rayleigh wave and the piezoelastic surface wave, theoretically discovered by Bleustein (1968) and Gulyaev (1969), are investigated in the context of a plane strain problem in elasticity and an antiplane problem in electroelasticity, respectively.

The developed models consist of hyperbolic equations for surface disturbances propagating with the finite speeds of the Rayleigh and Bleustein–Gulyaev (B-G) wave, together with elliptic equations for the interior domains. The sequence of Dirichlet or Neumann boundary-value problems is formulated in each of the cases. It is essential that both models support only surface singularities because of their mixed hyperbolic–elliptic type. This is in agreement with the general idea of the behaviour of a surface wave.

In Section 3 where we consider a Rayleigh wave, for the sake of simplicity, we consider separately the cases of normal and tangential impact loading. It is shown that the contribution of the Rayleigh wave induced by a vertical instantaneous point impulse is identical to that in the exact solution of the plane Lamb problem (Lamb, 1904; Poruchikov, 1993, pp. 139–149). This exact solution is presented in Appendix A. The B-G wave is considered in Section 4 for both a surface coated with a conducting electrode and an open surface. Again, a comparison with the exact solution of a specific 2D electroelastic problem is included (see Appendix B).

Finally, we remark that the method of constructing explicit asymptotic models is not restricted to elastic and piezoelastic solids. As another example, we cite the paper Ruderman (1992), which analysed the magnetohydrodynamic Alfvén wave.

2. Statement of the problem

One of the most famous examples of a surface wave is the elastic Rayleigh wave. Here we shall derive an approximate model for the plane strain problem.

Consider a linearly elastic isotropic half-plane, occupying the domain $-\infty < x < \infty$, $0 \leq y < \infty$. The governing equations of motion are taken in their classical form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \quad (2.1)$$

where ϕ and ψ are the wave potentials, and c_1 and c_2 are the longitudinal and transverse wave speeds, which may be presented in terms of Lamé moduli μ , λ , and the mass density ρ as

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}. \quad (2.2)$$

Mechanical displacement components can be obtained from the wave potentials as

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad u_2 = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}. \quad (2.3)$$

The boundary conditions on the surface $y = 0$ are

$$\sigma_{22}(x, 0, t) = -P_1, \quad \sigma_{12}(x, 0, t) = -P_2, \quad (2.4)$$

where $P_i = P_i(x, t)$, $i = 1, 2$.

The traction components are expressed as

$$\begin{aligned}\sigma_{22} &= \mu \left[(\kappa^2 - 2) \frac{\partial^2 \phi}{\partial x^2} + \kappa^2 \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \right], \\ \sigma_{12} &= \mu \left[2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right],\end{aligned}\tag{2.5}$$

with $\kappa = c_1/c_2$.

Here and below we consider only homogeneous initial data.

The self-similar solution in the variables $\zeta = x - c_R t$, y (where c_R is Rayleigh wave speed) was constructed for homogeneous problem (2.1), (2.5) with $\sigma_{22} = \sigma_{12} = 0$ in Friedlander (1948) and Chadwick (1976). To incorporate the effect of surface loading, we perturb below this self-similar solution in slow time $\tau = \varepsilon t$ ($\varepsilon \ll 1$), with small parameter expressing the deviation of a typical phase velocity from the Rayleigh one. For example, the surface ‘near-resonant’ excitation in the form $A \exp[ik(x - vt)]$ with the given amplitude A , wave number k , and the phase velocity v , dictates the small parameter $\varepsilon = v - c_R$, and can be expressed, therefore, in terms of variables ζ , τ as $A \exp[ik(\zeta - \tau)]$. This perturbation allows us to evaluate Rayleigh wave contribution into the overall dynamic response.

The same approach can be used in the case of a piezoelectric material where surface waves also exist for the antiplane problem (see Bleustein, 1968). Consider a transversely isotropic (e.g. crystal class C_{6mm}) piezoelectric half-plane with $y \geq 0$, with the out of plane axis being oriented in the direction of sixfold axis for a crystal in the class C_{6mm} . The governing equations can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,\tag{2.6}$$

where u denotes out of plane displacement component, ρ is mass density, $c_0 = \bar{c}_{44}/\rho$ is the low-frequency limit of the shear horizontal wave speed, $\bar{c}_{44} = c_{44} + e_{15}^2/\epsilon_{11}$ is piezoelectrically stiffened elastic constant and the function ψ is defined in terms of the electric potential ϕ as

$$\psi = \phi - \frac{e_{15}}{\epsilon_{11}} u,$$

where c_{44} , e_{15} and ϵ_{11} are material elastic, piezoelectric and dielectric constants, respectively.

Below, we concentrate on two types of boundary conditions imposed on the surface $y = 0$, namely,

- (i) surface completely coated with an infinitesimally thin perfectly conducting electrode which is grounded

$$\sigma_{23} = \bar{c}_{44} \frac{\partial u}{\partial y} + e_{15} \frac{\partial \psi}{\partial y} = -P, \quad \phi = \frac{e_{15}}{\epsilon_{11}} u + \psi = 0,\tag{2.7}$$

- (ii) surface in contact with a vacuum

$$\sigma_{23} = -P, \quad \phi = \hat{\phi}, \quad e_{15} \frac{\partial u}{\partial y} - \epsilon_{11} \frac{\partial \phi}{\partial y} = \frac{\partial \hat{\phi}}{\partial y},\tag{2.8}$$

where $P = P(x, t)$ is a mechanical load, σ_{23} is the appropriate component of the stress tensor and $\hat{\phi}$ is the electric potential in vacuum, satisfying

$$\nabla^2 \hat{\phi} = 0, \quad y \leq 0.$$

For more detail on formulation of piezoelectric problems, see Ikeda (1990).

3. Model for Rayleigh wave

The governing equations (2.1) may now be represented in terms of the new variables as

$$\begin{aligned} \frac{\partial^2 \phi}{\partial y^2} + \left(1 - \frac{c_R^2}{c_1^2}\right) \frac{\partial^2 \phi}{\partial \xi^2} + 2\varepsilon \frac{c_R}{c_1^2} \frac{\partial^2 \phi}{\partial \xi \partial \tau} - \varepsilon^2 \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial \tau^2} &= 0, \\ \frac{\partial^2 \psi}{\partial y^2} + \left(1 - \frac{c_R^2}{c_2^2}\right) \frac{\partial^2 \psi}{\partial \xi^2} + 2\varepsilon \frac{c_R}{c_2^2} \frac{\partial^2 \psi}{\partial \xi \partial \tau} - \varepsilon^2 \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial \tau^2} &= 0, \end{aligned} \tag{3.1}$$

The appropriate traction components can also be expressed as

$$\begin{aligned} \sigma_{22} &= \mu \left[(k^2 - 2) \frac{\partial^2 \phi}{\partial \xi^2} + k^2 \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial \xi \partial y} \right], \\ \sigma_{12} &= \mu \left[2 \frac{\partial^2 \phi}{\partial \xi \partial y} + \frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial^2 \psi}{\partial y^2} \right]. \end{aligned} \tag{3.2}$$

We now seek asymptotic solutions for the potentials in the dimensionless form

$$\begin{aligned} \phi &= \frac{P_*}{\varepsilon \mu} (\phi_0(\xi, y, \tau) + \varepsilon \phi_1(\xi, y, \tau) + \dots), \\ \psi &= \frac{P_*}{\varepsilon \mu} (\psi_0(\xi, y, \tau) + \varepsilon \psi_1(\xi, y, \tau) + \dots), \end{aligned} \tag{3.3}$$

where

$$P_* = \max_{x,t} (P_{1,2}(x, t)).$$

To leading order (3.1) is reduced to elliptic equations

$$\begin{aligned} \frac{\partial^2 \phi_0}{\partial y^2} + \left(1 - \frac{c_R^2}{c_1^2}\right) \frac{\partial^2 \phi_0}{\partial \xi^2} &= 0, \\ \frac{\partial^2 \psi_0}{\partial y^2} + \left(1 - \frac{c_R^2}{c_2^2}\right) \frac{\partial^2 \psi_0}{\partial \xi^2} &= 0. \end{aligned} \tag{3.4}$$

Following Friedlander (1948) and Chadwick (1976), we satisfy these equations with plane harmonic functions of the form

$$\phi_0 = \phi_0(\xi, k_1 y, \tau), \quad \psi_0 = \psi_0(\xi, k_2 y, \tau), \tag{3.5}$$

where

$$k_1^2 = 1 - \frac{c_R^2}{c_1^2}, \quad k_2^2 = 1 - \frac{c_R^2}{c_2^2}. \tag{3.6}$$

However, we assume that the functions in (3.5) may also contain the scaled time τ as a parameter.

For the ε -terms, we arrive at inhomogeneous equations

$$\begin{aligned}\frac{\partial^2 \phi_1}{\partial y^2} + k_1^2 \frac{\partial^2 \phi_1}{\partial \xi^2} &= -2 \frac{1 - k_1^2}{c_R} \frac{\partial^2 \phi_0}{\partial \xi \partial \tau}, \\ \frac{\partial^2 \psi_1}{\partial y^2} + k_2^2 \frac{\partial^2 \psi_1}{\partial \xi^2} &= -2 \frac{1 - k_2^2}{c_R} \frac{\partial^2 \psi_0}{\partial \xi \partial \tau}.\end{aligned}$$

The solution of these may be represented by

$$\phi_1 = \phi_{10} + y\phi_{11}, \quad \psi_1 = \psi_{10} + y\psi_{11}, \quad (3.7)$$

where ϕ_{10} , ϕ_{11} , ψ_{10} and ψ_{11} are harmonic functions.

For the functions ϕ_{11} and ψ_{11} , we have

$$\begin{aligned}\frac{\partial \phi_{11}}{\partial y} &= -\frac{1 - k_1^2}{c_R} \frac{\partial^2 \phi_0}{\partial \xi \partial \tau}, \\ \frac{\partial \psi_{11}}{\partial y} &= -\frac{1 - k_2^2}{c_R} \frac{\partial^2 \psi_0}{\partial \xi \partial \tau}.\end{aligned} \quad (3.8)$$

All the transformations below make use of the Cauchy–Riemann equations

$$\frac{\partial f}{\partial \xi} = \frac{1}{k} \frac{\partial f^*}{\partial y}, \quad \frac{\partial f}{\partial y} = -k \frac{\partial f^*}{\partial \xi}, \quad f^{**} = -f, \quad (3.9)$$

where $f = f(\xi, ky)$ is an arbitrary plane harmonic function and f^* denotes its harmonic conjugate.

In particular, by applying these identities to Formula (3.8), we rewrite the expansions (3.3) as

$$\begin{aligned}\phi &= \frac{P_*}{\varepsilon \mu} \left(\phi_0 + \varepsilon \left(\phi_{10} - y \frac{1 - k_1^2}{c_R k_1} \frac{\partial \phi_0^*}{\partial \tau} \right) \right), \\ \psi &= \frac{P_*}{\varepsilon \mu} \left(\psi_0 + \varepsilon \left(\psi_{10} - y \frac{1 - k_2^2}{c_R k_2} \frac{\partial \psi_0^*}{\partial \tau} \right) \right).\end{aligned} \quad (3.10)$$

It is convenient to separate the boundary conditions (2.4) into those for a normal load ($P_1 \neq 0$, $P_2 = 0$), and a tangential load ($P_1 = 0$, $P_2 \neq 0$).

In the case of a normal load, we have

$$\begin{aligned}2 \frac{\partial^2 \phi}{\partial \xi \partial y} + \frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial^2 \psi}{\partial y^2} &= 0, \\ (\kappa^2 - 2) \frac{\partial^2 \phi}{\partial \xi^2} + \kappa^2 \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial \xi \partial y} &= -\frac{P_1}{\mu}.\end{aligned} \quad (3.11)$$

First, substituting the solutions (3.10) into (3.11) we obtain to leading order

$$\begin{aligned} \left[2 \frac{\partial^2 \phi_0}{\partial y \partial \xi} + (1 + k_2^2) \frac{\partial^2 \psi_0}{\partial \xi^2} \right] \Big|_{y=0} &= 0, \\ \left[(1 + k_2^2) \frac{\partial^2 \phi_0}{\partial \xi^2} - 2 \frac{\partial^2 \psi_0}{\partial y \partial \xi} \right] \Big|_{y=0} &= 0. \end{aligned} \tag{3.12}$$

The first equation in (3.12) yields

$$\psi_0(\xi, 0, \tau) = \frac{2k_1}{1 + k_2^2} \phi_0^*(\xi, 0, \tau). \tag{3.13}$$

Then, the second gives

$$1 + k_2^2 - \frac{4k_1 k_2}{1 + k_2^2} = 0. \tag{3.14}$$

It can be easily seen, by using (3.6), that (3.14) may be rewritten as the classical Rayleigh equation

$$4 \sqrt{1 - \frac{c_R^2}{c_1^2}} \sqrt{1 - \frac{c_R^2}{c_2^2}} - \left(2 - \frac{c_R^2}{c_2^2} \right)^2 = 0. \tag{3.15}$$

To the next order, we have

$$\begin{aligned} \left[2 \frac{\partial^2 \phi_{10}}{\partial y \partial \xi} + (1 + k_2^2) \frac{\partial^2 \psi_{10}}{\partial \xi^2} - 2 \frac{1 - k_1^2}{c_R k_1} \frac{\partial^2 \phi_0^*}{\partial \xi \partial \tau} + 2 \frac{1 - k_2^2}{c_R k_2} \frac{\partial^2 \psi_0^*}{\partial y \partial \tau} \right] \Big|_{y=0} &= 0, \\ \left[(1 + k_2^2) \frac{\partial^2 \phi_{10}}{\partial \xi^2} - 2 \frac{\partial^2 \psi_{10}}{\partial \xi \partial y} + 2 \frac{1 - k_2^2}{c_R k_1} \frac{\partial^2 \phi_0^*}{\partial y \partial \tau} + 2 \frac{1 - k_2^2}{c_R k_2} \frac{\partial^2 \psi_0^*}{\partial \xi \partial \tau} \right] \Big|_{y=0} &= \frac{P_1}{P_*}. \end{aligned} \tag{3.16}$$

By using Relation (3.13), we can express the function ψ_{10} in the first equation of (3.16) as

$$\frac{\partial^2 \psi_{10}}{\partial \xi^2} \Big|_{y=0} = \frac{2}{1 + k_2^2} \left[k_1 \frac{\partial^2 \phi_{10}^*}{\partial \xi^2} + \frac{1}{c_R} \left(\frac{1 - k_1^2}{k_1} - 2k_1 \frac{1 - k_2^2}{1 + k_2^2} \right) \frac{\partial^2 \phi_0^*}{\partial \xi \partial \tau} \right] \Big|_{y=0}.$$

Then, substituting the derivative of ψ_{10} into the second equation, we get after lengthy but straightforward transformations

$$\left[\left(1 + k_2^2 - \frac{4k_1 k_2}{1 + k_2^2} \right) \frac{\partial^2 \phi_{10}}{\partial \xi^2} - \frac{4B}{1 + k_2^2} \frac{1}{c_R} \frac{\partial^2 \phi_0}{\partial \xi \partial \tau} \right] \Big|_{y=0} = \frac{P_1}{P_*}, \tag{3.17}$$

where

$$B = \frac{k_2}{k_1} (1 - k_1^2) + \frac{k_1}{k_2} (1 - k_2^2) - (1 - k_2^4). \tag{3.18}$$

From (3.14) we finally have

$$\frac{2\varepsilon}{c_R} \frac{\partial^2 \phi_a}{\partial \xi \partial \tau} \Big|_{y=0} = -\frac{1+k_2^2}{2\mu B} P_1, \quad (3.19)$$

where ϕ_a is an approximate solution

$$\phi_a = \frac{P_*}{\varepsilon\mu} \phi_0, \quad \psi_a = \frac{P_*}{\varepsilon\mu} \psi_0.$$

Changing variables, we can easily write down the leading order operator identity

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c_R^2} \frac{\partial^2}{\partial t^2} = \frac{2\varepsilon}{c_R} \frac{\partial^2}{\partial \xi \partial \tau}. \quad (3.20)$$

Then, transforming (3.19) to original variables, we obtain

$$\frac{\partial^2 \phi_s}{\partial x^2} - \frac{1}{c_R^2} \frac{\partial^2 \phi_s}{\partial t^2} = -\frac{1+k_2^2}{2\mu B} P_1, \quad (3.21)$$

where $\phi_s = \phi_a|_{y=0}$. The boundary condition for the second potential ψ can be found from (3.13) as

$$\frac{\partial \psi_a}{\partial x} \Big|_{y=0} = -\frac{2}{1+k_2^2} \frac{\partial \phi_a}{\partial y} \Big|_{y=0}. \quad (3.22)$$

This equation shows that the wave potentials are related to each other by means of a Hilbert transform, as has been shown in Chadwick (1976).

To obtain the interior field we use (3.4)

$$\begin{aligned} \frac{\partial^2 \phi_a}{\partial y^2} + k_1^2 \frac{\partial^2 \phi_a}{\partial x^2} &= 0, \\ \frac{\partial^2 \psi_a}{\partial y^2} + k_2^2 \frac{\partial^2 \psi_a}{\partial x^2} &= 0. \end{aligned} \quad (3.23)$$

Thus, we have derived an explicit asymptotic model consisting of a hyperbolic equation on the surface and elliptic equations for the interior; in doing so, the interior field may be found by solving a Dirichlet problem. It is remarkable that elliptic problems do not allow surface singularities to come through the interior. This is in complete agreement with the general idea of a surface wave.

In the case of a tangential load, the boundary conditions (2.4) become

$$\begin{aligned} (\kappa^2 - 2) \frac{\partial^2 \phi}{\partial \xi^2} + \kappa^2 \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial \xi \partial y} &= 0, \\ 2 \frac{\partial^2 \phi}{\partial \xi \partial y} + \frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial^2 \psi}{\partial y^2} &= -\frac{P_2}{\mu}. \end{aligned} \quad (3.24)$$

In the same manner we obtain, first, the leading order equations

$$\left[(1 + k_2^2) \frac{\partial^2 \phi_0}{\partial \xi^2} - 2 \frac{\partial^2 \psi_0}{\partial \xi \partial y} \right] \Big|_{y=0} = 0, \tag{3.25}$$

$$\left[2 \frac{\partial^2 \phi_0}{\partial y \partial \xi} + (1 + k_2^2) \frac{\partial^2 \psi_0}{\partial \xi^2} \right] \Big|_{y=0} = 0. \tag{3.26}$$

Next, we have the link between ϕ_0 and ψ_0 in the form

$$\phi_0(\xi, 0, \tau) = -\frac{2k_2}{1 + k_2^2} \psi_0^*(\xi, 0, \tau), \tag{3.27}$$

as well as the Rayleigh equation (3.15).

Transforming the next order equations, we obtain

$$\left[(1 + k_2^2) \frac{\partial^2 \phi_{10}}{\partial \xi^2} - 2 \frac{\partial^2 \psi_{10}}{\partial \xi \partial y} + 2 \frac{1 - k_2^2}{c_R k_1} \frac{\partial^2 \phi_0^*}{\partial y \partial \tau} + 2 \frac{1 - k_2^2}{c_R k_2} \frac{\partial^2 \psi_0^*}{\partial \xi \partial \tau} \right] \Big|_{y=0} = 0,$$

$$\left[2 \frac{\partial^2 \phi_{10}}{\partial y \partial \xi} - (1 + k_2^2) \frac{\partial^2 \psi_{10}}{\partial \xi^2} - 2 \frac{1 - k_1^2}{c_R k_1} \frac{\partial^2 \phi_0^*}{\partial \xi \partial \tau} + 2 \frac{1 - k_2^2}{c_R k_2} \frac{\partial^2 \psi_0^*}{\partial y \partial \tau} \right] \Big|_{y=0} = -\frac{P_2}{P_*},$$

then we arrive at a relation in terms of ψ_0 only, given by

$$\frac{2\varepsilon}{c_R} \frac{\partial^2 \psi_a}{\partial \xi \partial \tau} \Big|_{y=0} = \frac{1 + k_2^2}{2\mu B} P_2. \tag{3.28}$$

Finally, we have for the tangential load ($\psi_s = \psi_a|_{y=0}$)

$$\frac{\partial^2 \psi_s}{\partial x^2} - \frac{1}{c_R^2} \frac{\partial^2 \psi_s}{\partial t^2} = \frac{1 + k_2^2}{2\mu B} P_2, \tag{3.29}$$

with

$$\frac{\partial \phi_a}{\partial x} \Big|_{y=0} = \frac{2}{1 + k_2^2} \frac{\partial \psi_a}{\partial y} \Big|_{y=0}. \tag{3.30}$$

The latter is a similar problem to that obtained before for the normal load.

As an example, consider normal load in the form of a point instantaneous impulse

$$P_1(x, t) = P_0 \delta(x) \delta(t). \tag{3.31}$$

In this case, the 1D wave equation on the surface

$$\frac{\partial^2 \phi_s}{\partial x^2} - \frac{1}{c_R^2} \frac{\partial^2 \phi_s}{\partial t^2} = -\frac{1 + k_2^2}{2\mu B} P_0 \delta(x) \delta(t) \tag{3.32}$$

has a well-known solution (e.g. Zauderer, 1983, p. 473), which is given by

$$\phi_s = \frac{1 + k_2^2}{4\mu B} P_0 c_R (H(x + c_R t) - H(x - c_R t)), \quad (3.33)$$

where $H(x)$ is a unit step function. The elliptic problem (3.23) and (3.33) can also be solved easily (e.g. see calculations in the paper by Cole & Huth, 1958, for more detail)

$$\phi_a = \frac{1 + k_2^2}{4\mu B} \frac{P_0 c_R}{\pi} \left(\tan^{-1} \left(\frac{x + c_R t}{k_1 y} \right) - \tan^{-1} \left(\frac{x - c_R t}{k_1 y} \right) \right). \quad (3.34)$$

Next, using (3.22) we get

$$\psi_a = \frac{k_1}{4\mu B} \frac{P_0 c_R}{\pi} (\log((x + c_R t)^2 + k_2^2 y^2) - \log((x - c_R t)^2 + k_2^2 y^2)). \quad (3.35)$$

These results coincide with the well-known analytical solution of the classical plane Lamb problem (see Lamb, 1904) reproduced in Appendix A. As might be expected, the perturbation procedure developed in the paper allows exact evaluation of the residues associated with the Rayleigh wave (see (A.4)).

In 2D vicinities of surface wave-fronts ($x = \pm c_R t$), displacement and stress fields determined by (3.34), (3.35) dominate in the overall dynamic response of an elastic half-plane. It is also clear that the surface wave contribution will be dominant in the far field of δ -shaped distributed loads.

4. Model for Bleustein–Gulyaev wave

The governing equations (2.6) are rewritten in terms of new variables with $\zeta = x - c_{BG} t$ (where c_{BG} denotes the B-G surface wave speed) as

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} + \left(1 - \frac{c_{BG}^2}{c_0^2} \right) \frac{\partial^2 u}{\partial \zeta^2} + 2\varepsilon \frac{c_{BG}}{c_0^2} \frac{\partial^2 u}{\partial \zeta \partial \tau} - \varepsilon^2 \frac{1}{c_0^2} \frac{\partial^2 u}{\partial \tau^2} &= 0, \\ \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial \zeta^2} &= 0. \end{aligned} \quad (4.1)$$

As before we use the asymptotic solution

$$\begin{aligned} u &= \frac{P_*}{\varepsilon} (u_0(\zeta, y, \tau) + \varepsilon u_1(\zeta, y, \tau) + \dots), \\ \psi &= \frac{P_*}{\varepsilon} (\psi_0(\zeta, y, \tau) + \varepsilon \psi_1(\zeta, y, \tau) + \dots). \end{aligned} \quad (4.2)$$

First, we satisfy the governing equations to leading order by making use of the arbitrary plane harmonic functions

$$u_0 = u_0(\zeta, ky, \tau), \quad \psi_0 = \psi_0(\zeta, y, \tau), \quad (4.3)$$

with

$$k^2 = 1 - \frac{c_{BG}^2}{c_0^2}.$$

To the next order, we have (compare with (3.7))

$$\begin{aligned} u_1 &= u_{10} + yu_{11}, & \psi_1 &= \psi_1(\zeta, y, \tau), \\ u_{10} &= u_{10}(\zeta, ky, \tau), & u_{11} &= -\frac{1 - k^2}{c_{BG}k} \frac{\partial u_0^*}{\partial \tau}, \end{aligned} \tag{4.4}$$

where u_{10} and ψ_1 are plane harmonic functions.

In the case of an electrode coated surface, the original boundary conditions can be rewritten as

$$\begin{aligned} \bar{c}_{44} \frac{\partial u}{\partial y} + e_{15} \frac{\partial \psi}{\partial y} &= -P, \\ \frac{e_{15}}{\epsilon_{11}} u + \psi &= 0. \end{aligned} \tag{4.5}$$

Inserting the solution (4.2)–(4.4) into (4.5), we can easily obtain

$$\begin{aligned} \psi_0 &= -\frac{e_{15}}{\epsilon_{11}} u_0, & \psi_1 &= -\frac{e_{15}}{\epsilon_{11}} u_{10}, \\ \left(k - \frac{e_{15}}{\epsilon_{11} \bar{c}_{44}}\right) \frac{\partial u_0}{\partial y} + \epsilon \left(k - \frac{e_{15}}{\epsilon_{11} \bar{c}_{44}}\right) \frac{\partial u_{10}}{\partial y} - \epsilon \frac{1 - k^2}{c_{BG}} \frac{\partial u_0^*}{\partial \tau} &= -\epsilon \frac{k}{\bar{c}_{44}} \frac{P}{P_*}. \end{aligned} \tag{4.6}$$

The coefficient of $\partial u_0 / \partial y$ vanishes provided that

$$c_{BG}^2 = \frac{\bar{c}_{44}}{\rho} (1 - k^2), \tag{4.7}$$

with

$$k = \frac{e_{15}^2}{\epsilon_{11} \bar{c}_{44}}, \tag{4.8}$$

where the parameter k coincides with the electromechanical coupling factor. As might be expected, (4.7) with (4.8) gives the exact B-G wave speed (Bleustein, 1968).

Analogous to what has been done in Section 3, we utilize (3.9) to transform the second equation in (4.6). The result is

$$\frac{2\epsilon}{c_{BG}} \frac{\partial^3 u_a}{\partial \zeta \partial y \partial \tau} = -\frac{2k^2}{\rho c_{BG}^2} \frac{\partial^2 P}{\partial \zeta^2}, \tag{4.9}$$

with

$$u_a = \frac{P_*}{\epsilon} u_0.$$

For a surface in contact with a vacuum, we present the boundary conditions (2.8) as

$$\begin{aligned}\bar{c}_{44} \frac{\partial u}{\partial y} + e_{15} \frac{\partial \psi}{\partial y} &= -P, \\ \frac{e_{15}}{\epsilon_{11}} u + \psi &= \hat{\phi}, \\ -\epsilon_{11} \frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\phi}}{\partial y},\end{aligned}\tag{4.10}$$

where $\hat{\phi}$ is the electric potential in the vacuum which satisfies

$$\nabla^2 \hat{\phi} = 0 \quad (y \leq 0).$$

The solution of the latter may be expanded as

$$\hat{\phi} = \frac{P_*}{\epsilon} (\hat{\phi}_0 + \varepsilon \hat{\phi}_1 + \dots),$$

where $\hat{\phi}_0 = \hat{\phi}_0(\xi, y, \tau)$, $\hat{\phi}_1 = \hat{\phi}_1(\xi, y, \tau)$ are arbitrary plane harmonic functions.

Next, by transforming the boundary conditions (4.10) we have the same result as that for a coated surface, except that the coupling factor k now takes the value

$$k = \frac{e_{15}^2}{\epsilon_{11} \bar{c}_{44} (1 + \epsilon_{11})}.\tag{4.11}$$

Utilizing (3.20), we obtain the 1D hyperbolic equation

$$\frac{\partial^2 \chi_s}{\partial x^2} - \frac{1}{c_{\text{BG}}^2} \frac{\partial^2 \chi_s}{\partial t^2} = -\frac{2k^2}{\rho c_{\text{BG}}^2} \frac{\partial^2 P}{\partial x^2},\tag{4.12}$$

with

$$\chi_s = \left. \frac{\partial u_a}{\partial y} \right|_{y=0}.$$

By neglecting $O(\varepsilon)$ terms in (4.1), we arrive at the elliptic equation

$$\frac{\partial^2 u_a}{\partial y^2} + k^2 \frac{\partial^2 u_a}{\partial x^2} = 0.\tag{4.13}$$

The solution of the wave equation (4.12) provides a Neumann boundary condition for the elliptic equation (4.13). This problem corresponds to the wave field over the half-plane interior.

The electric field can be derived from the Dirichlet problem

$$\nabla^2 \psi_a = 0,\tag{4.14}$$

with

$$\psi_a|_{y=0} = -\frac{e_{15}}{\epsilon_{11}}u_a \Big|_{y=0}. \tag{4.15}$$

In conclusion, consider the effect of a mechanical load applied to a coated surface. Let us set in (4.12)

$$P(x, t) = \delta(t)F(x), \tag{4.16}$$

where $F(x)$ is a concentrated function, e.g.

$$F(x) = \frac{1}{\sqrt{2\pi}L} \exp\left(-\frac{x^2}{2L^2}\right). \tag{4.17}$$

First, we solve the elliptic equations (4.13) and (4.14) using the standard Fourier transform technique

$$\begin{aligned} \frac{d^2U_0^F}{dy^2} &= k^2 p^2 U_0^F, \\ \frac{d^2\Psi_0^F}{dy^2} &= p^2 \Psi_0^F, \end{aligned} \tag{4.18}$$

where the superscript ‘‘F’’ denotes the Fourier transform with respect to the variable x and p is the transform parameter. Then

$$\begin{aligned} U_0^F &= U_s^F \exp(-kpy), \\ \Psi_0^F &= \Psi_s^F \exp(-py). \end{aligned} \tag{4.19}$$

In order to solve (4.12), we apply the Laplace transform with respect to time t , giving

$$\chi_s^{FL} = -kpU_s^{FL} = -\frac{2k^2}{\rho} \frac{p^2 F^F}{c_{BG}^2 p^2 + s^2},$$

where the superscript ‘‘L’’ denotes the Laplace transform and variable s is the transform parameter.

The Laplace transform can be easily inverted

$$U_s^F = -\frac{ik}{\rho c_{BG}} F^F (e^{ic_{BG}pt} - e^{-ic_{BG}pt}).$$

Finally, for the transformed displacement, we get

$$U_0^F = -\frac{ik}{\rho c_{BG}} F^F e^{-kyp} (e^{ipc_{BG}t} - e^{-ipc_{BG}t}), \tag{4.20}$$

whereas the electric potential can be found from (4.15) and (4.19)

$$\Phi_0^F = -\frac{ie_{15}k}{\epsilon_{11}\rho c_{BG}} F^F (e^{-kpy} - e^{-py})(e^{ipc_{BG}t} - e^{-ipc_{BG}t}). \tag{4.21}$$

By inverting the Fourier transform on the surface, we arrive at the classical wave solution

$$u_s = \frac{k}{\rho c_{BG}} (F(x - c_{BG}t) - F(x + c_{BG}t)).$$

5. Concluding remarks

In this paper, a methodology is developed for the analysis of elastic and electroelastic surface waves. It results in the derivation of explicit models demonstrating the dual hyperbolic–elliptic nature of a surface wave. In fact, both of the formulated models consist of a hyperbolic equation describing wave propagation along the surface (see (3.21), (3.29) and (4.12)) with the Rayleigh or B-G wave speed, and two elliptic equations for the interior; in doing so, each of these problems can be treated separately. It is clear that in the framework of these models, singularities may arise only on the surface.

The proposed approach not only is restricted to the surface waves investigated in the paper but also allows various generalizations including those for 3D problems, anisotropic and prestressed bodies, curved surfaces, and interfacial waves. It is important that for some of these problems, existence and uniqueness have been already investigated (e.g. see Mielke & Fu, 2004).

The proposed models can be especially useful for the solution of the problems with a major contribution involving surface wave phenomena. Among these, we mention analysis of the vicinities of surface wave-fronts in far field as well as investigation of various resonant phenomena. The latter involves in particular the resonant effect of moving loads on an elastic half-space (e.g. see Gol'dshtein, 1965; Freund, 1973).

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Appendix

Appendix A. Exact solution of the plane Lamb problem

Let us extract the Rayleigh wave contribution from the exact solution of the plane Lamb problem in the case of normal instantaneous point impulse (see Poruchikov, 1993, pp. 139–149, for more detail)

$$P_1(x, t) = P_0\delta(x)\delta(t). \tag{A.1}$$

In terms of the double Fourier–Laplace transforms, the solution of the problems (2.1) and (2.4) becomes

$$\begin{aligned} \Phi^{\text{FL}} &= \sqrt{\frac{2}{\pi}} \frac{P_0}{\mu} \frac{1 + K_2^2}{4K_1K_2 - (1 + K_2^2)^2} \frac{\exp(-K_1py)}{p^2}, \\ \Psi^{\text{FL}} &= \sqrt{\frac{2}{\pi}} \frac{P_0}{\mu} \frac{2iK_1}{4K_1K_2 - (1 + K_2^2)^2} \frac{\exp(-K_2py)}{p^2}, \end{aligned} \tag{A.2}$$

with

$$K_1^2 = 1 + \frac{s^2}{p^2c_1^2}, \quad K_2^2 = 1 + \frac{s^2}{p^2c_2^2},$$

where

$$\begin{aligned} \Phi^{\text{FL}}(p, y, s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \phi(x, y, t) e^{-st} \cos(px) dt dx, \\ \Psi^{\text{FL}}(p, y, s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \psi(x, y, t) e^{-st} \sin(px) dt dx. \end{aligned} \tag{A.3}$$

The Rayleigh wave can be extracted from the latter by taking residues at the poles $s = \pm ic_R p$, which gives for Fourier transforms

$$\begin{aligned} \Phi^{\text{F}} &= \sqrt{\frac{2}{\pi}} \frac{P_0c_R}{\mu} \frac{1 + k_2^2}{2B} \frac{\exp(-k_1py)}{p} \sin(c_R pt), \\ \Psi^{\text{F}} &= \sqrt{\frac{2}{\pi}} \frac{P_0c_R}{\mu} \frac{ik_1}{B} \frac{\exp(-k_2py)}{p} \sin(c_R pt), \end{aligned} \tag{A.4}$$

where constants k_1, k_2, B are defined by (3.6) and (3.18), respectively.

By calculating inverse Fourier transforms, we immediately arrive at the approximate solutions (3.34) and (3.35).

Appendix B. Exact solution for the example in Section 3

We apply the double integral transform technique, setting

$$\begin{aligned} U^{\text{FL}}(p, y, s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty u(x, y, t) e^{-st} \cos(px) dt dx, \\ \Psi^{\text{FL}}(p, y, s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \psi(x, y, t) e^{-st} \cos(px) dt dx. \end{aligned} \quad (\text{B.1})$$

The transformed equations (2.6) become

$$\begin{aligned} \frac{d^2 U^{\text{FL}}}{dy^2} &= K^2 p^2 U^{\text{FL}}, \\ \frac{d^2 \Psi^{\text{FL}}}{dy^2} &= p^2 \Psi^{\text{FL}}, \end{aligned} \quad (\text{B.2})$$

with

$$K^2 = 1 + \frac{s^2}{p^2 c_0^2}.$$

By inserting

$$\begin{aligned} U^{\text{FL}} &= U_s^{\text{FL}} \exp(-Kpy), \\ \Psi^{\text{FL}} &= \Psi_s^{\text{FL}} \exp(-py) \end{aligned} \quad (\text{B.3})$$

into the boundary conditions (2.7), we have

$$U_s^{\text{FL}} = \frac{1}{\rho} F^{\text{F}} \frac{Kp + kp}{s^2 + p^2 c_{\text{BG}}^2}, \quad (\text{B.4})$$

$$\Psi_s^{\text{FL}} = -\frac{e_{15}}{\epsilon_{11}} U_s^{\text{FL}}, \quad (\text{B.5})$$

where k is defined by (4.8).

The contribution of the B-G wave is represented by the sum of residues corresponding to the poles $s = \pm ic_{\text{BG}}p$. Thus

$$U_{\text{BG}}^{\text{F}} = -\frac{ik}{\rho c_{\text{BG}}} F^{\text{F}} e^{-kyp} (e^{ipc_{\text{BG}}t} - e^{-ipc_{\text{BG}}t}). \quad (\text{B.6})$$

We can also easily obtain

$$\Phi_{\text{BG}}^{\text{F}} = -\frac{ie_{15}k}{\epsilon_{11}\rho c_{\text{BG}}} F^{\text{F}} (e^{-kpy} - e^{-py})(e^{ipc_{\text{BG}}t} - e^{-ipc_{\text{BG}}t}). \quad (\text{B.7})$$

Both these transforms coincide with those in the case of the approximate solutions (4.20) and (4.21).