Appendix 5

Tables of Laplace Transforms

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Many Laplace transforms and their inverses have been derived and compiled. Extensive compilations of Laplace transforms are available from many sources, including entire books (see for example, Oberhettinger and Badii, 1973). In this appendix we reproduce the Laplace transforms compiled in Churchill's excellent textbook **Operational Mathematics** (3rd edition, 1972). These transforms are supplemented with transforms compiled by Carslaw and Jaeger (1959) and Hantush (1964). The transforms compiled in Carslaw and Jaeger (1959) [**Conduction of Heat in Solids,** 2nd edition, 1959] arise frequently in solute transport applications. M.S. Hantush included a short table of Laplace transforms in his landmark publication *Hydraulics of Wells* (1964). These transforms are also reproduced here because the transforms are of immediate application to solutions for aquifer tests.

1. Laplace Transform Operations (adapted from Churchill, 1972)

	$\bar{f}(p)$ $(\mathrm{Re}\ p > \alpha)$	**F(t)** $(t > 0)$
1	$\int_0^\infty e^{-pt} F(t)\,dt = L\{F\}$	$F(t)$
2	$\bar{f}(p)$	$\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{tz}\bar{f}(z)\,dz$
3	$p\bar{f}(p) - F(0)$	$F'(t) \equiv \frac{\partial F}{\partial t}$
4	$p^n\bar{f}(p) - p^{n-1}F(0) - p^{n-2}F'(0) - \ldots - F^{(n-1)}(0)$	$F^{(n)}(t) \equiv \frac{\partial^n F}{\partial t^n}$
5	$\frac{1}{p}\bar{f}(p)$	$\int_0^t F(\tau)\,d\tau$
6	$e^{-ap}\bar{f}(p)$ $(a > 0)$	$F(t-a)$ if $t > a$ 0 if $t < a$
7	$\bar{f}(p-a)$	$e^{at}F(t)$ a = constant SHIFT THEOREM
8	$\bar{f}(p)\bar{g}(p)$	$\int_0^t F(\tau)G(t-\tau)\,d\tau = \int_0^t F(t-\tau)G(\tau)\,d\tau$ CONVOLUTION
9	$\bar{f}'(p)$	$-tF(t)$
10	$\bar{f}^{(n)}(p)$	$(-1)^n t^n F(t)$
11	$\int_p^\infty \bar{f}(x)\,dx$	$\frac{1}{t}F(t)$
12	$\bar{f}(cp)$ $(c > 0)$	$\frac{1}{c}F\left(\frac{t}{c}\right)$
13	$\frac{\int_0^a e^{-pt}F(t)\,dt}{1-e^{-ap}}$ $(a > 0)$	$F(t)$ if $F(t+a) = F(t)$
14	$\frac{\int_0^a e^{-pt}F(t)\,dt}{1+e^{-ap}}$ $(a > 0)$	$F(t)$ if $F(t+a) = -F(t)$
15	$\sum_{n=0}^\infty \frac{a_n}{p^{n+k}}$ $(k > 0)$	$\sum_{n=0}^\infty \frac{a_n}{\Gamma(n+k)} t^{n+k-1}$

2. Laplace Transforms (adapted from Churchill, 1972)

	$\bar{f}(p)$	$\boldsymbol{F(t)}\ (t>0)$
1	$\frac{1}{p}$	1
2	$\frac{1}{p^2}$	t
3	$\frac{1}{p^n}\quad (n=1,2,\ldots)$	$\frac{t^{n-1}}{(n-1)!}$ Hantush (1964; #2)
3a	$\frac{1}{p^{v+1}}\quad (v>-1)$	$\frac{t^v}{\Gamma(v+1)}$ Carslaw and Jaeger (1959; #2)
4	$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi t}}$
5	$\frac{1}{p\sqrt{p}}$	$2\sqrt{\frac{t}{\pi}}$
6	$p^{-\left(n+\frac{1}{2}\right)}\quad (n=1,2,\ldots)$	$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\ldots\cdot(2n-1)\sqrt{\pi}}$
7	$\frac{\Gamma(k)}{p^k}\quad (k>0)$	t^{k-1}
8	$\frac{1}{p-a}$	e^{at}
9	$\frac{1}{(p-a)^2}$	te^{at}
10	$\frac{1}{(p-a)^n}\quad (n=1,2,\ldots)$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
11	$\frac{\Gamma(k)}{(p-a)^k}\quad (k>0)$	$t^{k-1}e^{at}$
12	$\frac{1}{(p-a)(p-b)}$	$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$
13	$\frac{p}{(p-a)(p-b)}$	$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$
14	$\frac{1}{(p-a)(p-b)(p-c)}$	$-\frac{(b-c)e^{at}+(c-a)e^{bt}+(a-b)e^{ct}}{(a-b)(b-c)(c-a)}$

	$\bar{f}(p)$	**$F(t)$** $(t>0)$
15	$\dfrac{a}{p^2+a^2}$	$\sin at$
16	$\dfrac{p}{p^2+a^2}$	$\cos at$
17	$\dfrac{1}{p^2-a^2}$	$\dfrac{1}{a}\sinh at$
18	$\dfrac{p}{p^2-a^2}$	$\cosh at$
19	$\dfrac{1}{p\left(p^2+a^2\right)}$	$\dfrac{1}{a^2}(1-\cos at)$
20	$\dfrac{1}{p^2\left(p^2+a^2\right)}$	$\dfrac{1}{a^3}(at-\sin at)$
21	$\dfrac{1}{\left(p^2+a^2\right)^2}$	$\dfrac{1}{2a^3}(\sin at-at\cos at)$
22	$\dfrac{p}{\left(p^2+a^2\right)^2}$	$\dfrac{t}{2a}\sin at$
23	$\dfrac{p^2}{\left(p^2+a^2\right)^2}$	$\dfrac{1}{2a}(\sin at+at\cos at)$
24	$\dfrac{p^2-a^2}{\left(p^2+a^2\right)^2}$	$t\cos at$
25	$\dfrac{p}{\left(p^2+a^2\right)\left(p^2+b^2\right)} \quad \left(a^2\neq b^2\right)$	$\dfrac{\cos at-\cos bt}{b^2-a^2}$
26	$\dfrac{1}{(p-a)^2+b^2}$	$\dfrac{1}{b}e^{at}\sin bt$
27	$\dfrac{p-a}{(p-a)^2+b^2}$	$e^{at}\cos bt$
28	$\dfrac{3a^2}{p^3+a^3}$	$e^{-at}-e^{\frac{at}{2}}\left(\cos\dfrac{at\sqrt{3}}{2}-\sqrt{3}\sin\dfrac{at\sqrt{3}}{2}\right)$
29	$\dfrac{4a^3}{p^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$
30	$\dfrac{p}{p^4+4a^4}$	$\dfrac{1}{2a^2}\sin at\sinh at$

	$\overline{f}(p)$	**F(t)** $(t>0)$
31	$\dfrac{1}{p^4-a^4}$	$\dfrac{1}{2a^3}(\sinh at-\sin at)$
32	$\dfrac{p}{p^4-a^4}$	$\dfrac{1}{2a^2}(\cosh at-\cos at)$
33	$\dfrac{8a^3p^2}{\left(p^2+a^2\right)^3}$	$\left(1+a^2t^2\right)\sin at-at\cos at$
34	$\dfrac{1}{p}\left(\dfrac{p-1}{p}\right)^n$	$L_n(t)=\dfrac{e^t}{n!}\dfrac{d^n}{dt^n}\left(t^ne^{-t}\right)$ $L_n(t)$ is the Laguerre polynomial of degree n
35	$\dfrac{p}{(p-a)\sqrt{p-a}}$	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
36	$\sqrt{p-a}-\sqrt{p-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}\left(e^{bt}-e^{at}\right)$
37	$\dfrac{1}{\sqrt{p+a}}$	$\dfrac{1}{\sqrt{\pi t}}-ae^{a^2t}erfc\left(a\sqrt{t}\right)$
38	$\dfrac{\sqrt{p}}{p-a^2}$	$\dfrac{1}{\sqrt{\pi t}}+ae^{a^2t}erf\left(a\sqrt{t}\right)$
39	$\dfrac{\sqrt{p}}{p+a^2}$	$\dfrac{1}{\sqrt{\pi t}}-\dfrac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}\exp\left(\lambda^2\right)d\lambda$
40	$\dfrac{1}{\sqrt{p}\left(p-a^2\right)}$	$\dfrac{1}{a}e^{a^2t}erf\left(a\sqrt{t}\right)$
41	$\dfrac{1}{\sqrt{p}\left(p+a^2\right)}$	$\dfrac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}\,d\lambda$
42	$\dfrac{b^2-a^2}{\left(p-a^2\right)\left(b+\sqrt{p}\right)}$	$e^{a^2t}\left[b-a\,erf\left(a\sqrt{t}\right)\right]-b\,e^{b^2t}erfc\left(b\sqrt{t}\right)$
43	$\dfrac{1}{\sqrt{p}\left(\sqrt{p}+a\right)}$	$e^{a^2t}erfc\left(a\sqrt{t}\right)$
44	$\dfrac{1}{(p+a)\sqrt{p+b}}$	$\dfrac{1}{\sqrt{b-a}}e^{-at}erf\left(\sqrt{b-a}\sqrt{t}\right)$
45	$\dfrac{b^2-a^2}{\sqrt{p}\left(p-a^2\right)\left(\sqrt{p}+b\right)}$	$e^{a^2t}\left[\dfrac{b}{a}erf\left(a\sqrt{t}\right)-1\right]+e^{b^2t}erfc\left(b\sqrt{t}\right)$

	$\overline{f}(p)$	**F(t)** $(t>0)$
46	$\dfrac{(1-p)^n}{p^{n+\frac{1}{2}}}$	$\dfrac{n!}{(2n)!\sqrt{\pi t}}H_{2n}\left(\sqrt{t}\right)$ $H_n(x)$ is the Hermite polynomial $H_n(x)=e^{x^2}\left(\dfrac{d^n}{dx^n}\right)\left(e^{-x^2}\right)$
47	$\dfrac{(1-p)}{p^{n+1}\sqrt{p}}$	$-\dfrac{n!}{\sqrt{\pi}(2n+1)!}H_{2n+1}\left(\sqrt{t}\right)$
48	$\dfrac{\sqrt{p+2a}}{\sqrt{p}}-1$	$ae^{-at}\left[I_1(at)+I_0(at)\right]$ $I_n(x)=I^{-n}J_n(Ix)$ where J_n is Bessel's function of the first kind
49	$\dfrac{1}{\sqrt{p+a}\sqrt{p+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0\left(\dfrac{a-b}{2}t\right)$
50	$\dfrac{\Gamma(k)}{(p+a)^4(p+b)^4}\quad(k>0)$	$\sqrt{\pi}\left(\dfrac{t}{a-b}\right)^{k-\frac{1}{2}}e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\dfrac{a-b}{2}t\right)$
51	$\dfrac{1}{\sqrt{p+a}\sqrt{p+b}(p+b)}$	$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\dfrac{a-b}{2}t\right)+I_1\left(\dfrac{a-b}{2}t\right)\right]$
52	$\dfrac{\sqrt{p+2a}-\sqrt{p}}{\sqrt{p+2a}+\sqrt{p}}$	$\dfrac{1}{t}e^{-at}I_1(at)$
53	$\dfrac{(a-b)^4}{\left(\sqrt{p+a}+\sqrt{p+b}\right)^{2k}}\quad(k>0)$	$\dfrac{k}{t}e^{-\frac{1}{2}(a+b)t}I_k\left(\dfrac{a-b}{2}t\right)$
54	$\dfrac{\left(\sqrt{p+a}+\sqrt{p}\right)^{-2v}}{\sqrt{p}\sqrt{p+a}}\quad(v>-1)$	$\dfrac{1}{a^v}e^{-\frac{1}{2}at}I_v\left(\dfrac{1}{2}at\right)$
55	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(at)$
56	$\dfrac{\left(\sqrt{p^2+a^2}-p\right)^v}{\sqrt{p^2+a^2}}\quad(v>-1)$	$a^vJ_v(at)$
57	$\dfrac{1}{\left(p^2+a^2\right)^k}\quad(k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$
58	$\left(\sqrt{p^2+a^2}-p\right)^k\quad(k>0)$	$\dfrac{ka^k}{t}J_k(at)$

	$\bar{f}(p)$	**F(t)** $(t>0)$
59	$\dfrac{\left(p-\sqrt{p^2-a^2}\right)^v}{\sqrt{p^2-a^2}} \quad (v>-1)$	$a^v I_v(at)$
60	$\dfrac{1}{\left(s^2-a^2\right)^k} \quad (k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$
61	$\dfrac{e^{-kp}}{p}$	$H(t-k)=\begin{cases}0 & \text{when } 0<t<k \\ 1 & \text{when } t>k\end{cases}$ Heaviside step function
62	$\dfrac{e^{-kp}}{p^2}$	$=0 \quad$ when $0<t<k$ $=t-k$ when $t>k$
63	$\dfrac{e^{-kp}}{p^\mu} \quad (\mu>0)$	$=0 \quad$ when $0<t<k$ $=\dfrac{(t-k)^{\mu-1}}{\Gamma(\mu)}$ when $t>k$
64	$\dfrac{1-e^{-kp}}{p}$	$=1$ when $0<t<k$ $=0$ when $t>k$
65	$\dfrac{1}{p\left(1-e^{-kp}\right)}=\dfrac{1+\coth\frac{1}{2}kp}{2p}$	$1+\left[\dfrac{t}{k}\right]=n$ when $(n-1)k<t<nk \quad (n=1,2,\ldots)$
66	$\dfrac{1}{p\left(e^{kp}-a\right)}$	$=0 \quad$ when $0<t<k$ $=1+a+a^2+\ldots+a^{n-1}$ when $nk<t<(n+1)k \quad (n=1,2,\ldots)$
67	$\dfrac{1}{p}\tanh kp$	$M(2k,t)=(-1)^{n-1}$ when $2k(n-1)<t<2kn \quad (n=1,2,\ldots)$
68	$\dfrac{1}{p\left(1+e^{-kp}\right)}$	$\dfrac{1}{2}M(k,t)+\dfrac{1}{2}=\dfrac{1-(-1)^n}{2}$ when $(n-1)k<t<nk$
69	$\dfrac{1}{p^2}\tanh kp$	$H(2k,t)$
70	$\dfrac{1}{p\sinh kp}$	$F(t)=2(n-1)$ when $(2n-3)k<t<(2n-1)k \quad (t>0)$
71	$\dfrac{1}{p\cosh kp}$	$M(2k,t+3k)+1=1+(-1)^n$ when $(2n-3)k<t<(2n-1)k \quad (t>0)$
72	$\dfrac{1}{p}\coth kp$	$F(t)=2n-1$ when $2k(n-1)<t<2kn$
73	$\dfrac{k}{p^2+k^2}\coth\dfrac{\pi p}{2k}$	$\left\|\sin kt\right\|$

	$\overline{f}(p)$	**F(t)** $(t>0)$
74	$\dfrac{1}{(p^2+1)(1-e^{-\pi p})}$	$\frac{1}{2}(\sin t+\lvert\sin t\rvert)$
75	$\frac{1}{p}e^{-\frac{k}{p}}$	$J_0(2\sqrt{kt})$
76	$\frac{1}{\sqrt{p}}e^{-\frac{k}{p}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
77	$\frac{1}{\sqrt{p}}e^{\frac{k}{p}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$
78	$\frac{1}{p\sqrt{p}}e^{-\frac{k}{p}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$
79	$\frac{1}{p\sqrt{p}}e^{\frac{k}{p}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$
80	$\frac{1}{p^{\mu}}e^{-\frac{k}{p}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{(\mu-1)}{2}}J_{\mu-1}(2\sqrt{kt})$
81	$\frac{1}{p^{\mu}}e^{\frac{k}{p}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{(\mu-1)}{2}}I_{\mu-1}(2\sqrt{kt})$
82	$e^{-k\sqrt{p}}\quad(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$
83	$\frac{1}{p}e^{-k\sqrt{p}}\quad(k\ge 0)$	$erfc\left(\frac{k}{2\sqrt{t}}\right)$
83A	$\frac{1}{p-a}e^{-k\sqrt{p}}\quad(a>0)$	$\frac{1}{2}e^{at}\left[e^{-k\sqrt{a}}erfc\left\{\frac{k}{2\sqrt{t}}-\sqrt{at}\right\}+e^{k\sqrt{a}}erfc\left\{\frac{k}{2\sqrt{t}}+\sqrt{at}\right\}\right]$
84	$\frac{1}{\sqrt{p}}e^{-k\sqrt{p}}\quad(k\ge 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$
85	$\frac{1}{p\sqrt{p}}e^{-k\sqrt{p}}\quad(k\ge 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\ erfc\left(\frac{k}{2\sqrt{t}}\right)$
86	$\frac{ae^{-k\sqrt{p}}}{s(a+\sqrt{p})}\quad(k\ge 0)$	$-e^{ak}e^{a^2t}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+erfc\left(\frac{k}{2\sqrt{t}}\right)$
87	$\frac{e^{-k\sqrt{p}}}{\sqrt{p}(a+\sqrt{p})}\quad(k\ge 0)$	$e^{ak}e^{a^2t}erfc\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$

	$\bar{f}(p)$	**F(t)** $(t>0)$
88	$\dfrac{e^{-k\sqrt{p(p+a)}}}{\sqrt{p(p+a)}}$	$=0$ when $0<t<k$ $=e^{-\frac{1}{2}at}I_0\left(\frac{1}{2}a\sqrt{t^2-k^2}\right)$ when $t>k$
89	$\dfrac{e^{-k\sqrt{p^2+a^2}}}{\sqrt{p^2+a^2}}$	$=0$ when $0<t<k$ $=J_0\left(a\sqrt{t^2-k^2}\right)$ when $t>k$
90	$\dfrac{e^{-k\sqrt{p^2-a^2}}}{\sqrt{p^2-a^2}}$	$=0$ when $0<t<k$ $=I_0\left(a\sqrt{t^2-k^2}\right)$ when $t>k$
91	$\dfrac{e^{-k\left(\sqrt{p^2+a^2}-p\right)}}{\sqrt{p^2+a^2}}\quad (k\geq 0)$	$J_0\left(a\sqrt{t^2+2kt}\right)$
92	$e^{-kp}-e^{-k\sqrt{p^2+a^2}}$	$=0$ when $0<t<k$ $=\dfrac{ak}{\sqrt{t^2-k^2}}J_1\left(a\sqrt{t^2-k^2}\right)$ when $t>k$
93	$e^{-k\sqrt{p^2-a^2}}-e^{-kp}$	$=0$ when $0<t<k$ $=\dfrac{ak}{\sqrt{t^2-k^2}}I_1\left(a\sqrt{t^2-k^2}\right)$ when $t>k$
94	$\dfrac{a^{v}e^{-k\sqrt{p^2+a^2}}}{\sqrt{p^2+a^2}\left(\sqrt{p^2+a^2}+p\right)^{v}}\quad (v>-1)$	$=0$ when $0<t<k$ $=\left(\dfrac{t-k}{t+k}\right)^{\frac{1}{2}v}J_v\left(a\sqrt{t^2-k^2}\right)$ when $t>k$
95	$\dfrac{1}{p}\ln p$	$\Gamma'(1)-\ln t\quad \left[\Gamma'(1)=-0.5772\right]$
96	$\dfrac{1}{p^k}\ln p\quad (k>0)$	$t^{k-1}\left\{\dfrac{\Gamma'(k)}{\left[\Gamma(k)\right]^2}-\dfrac{\ln t}{\Gamma(k)}\right\}$
97	$\dfrac{\ln p}{p-a}\quad (a>0)$	$e^{at}\left[\ln a+E_1(at)\right]$ $E_1(t)$ is the exponential-integral function
98	$\dfrac{\ln p}{p^2+1}$	cos t Si t – sin t Ci t Si is the sine-integral function Ci is the cosine-integral function
99	$\dfrac{p\ln p}{p^2+1}$	sin t Si t –cos t Ci t
100	$\dfrac{1}{p}\ln(1+kp)\quad (k>0)$	$E_1\left(\dfrac{t}{k}\right)$
101	$\ln\dfrac{p-a}{p-b}$	$\dfrac{1}{t}\left(e^{bt}-e^{at}\right)$

	$\bar{f}(p)$	**F(t)** $(t>0)$
102	$\frac{1}{p}\ln\left(1+k^2p^2\right)$	$-2Ci\left(\frac{t}{k}\right)$
103	$\frac{1}{p}\ln\left(p^2+a^2\right) \quad (a>0)$	$2\ \ln a-2\,\text{Ci}(at)$
104	$\frac{1}{p^2}\ln\left(p^2+a^2\right) \quad (a>0)$	$\frac{2}{a}\left[at\ln a+\sin at-at\ \text{Ci}(at)\right]$
105	$\ln\frac{p^2+a^2}{p^2}$	$\frac{2}{t}(1-\cos at)$
106	$\ln\frac{p^2-a^2}{p^2}$	$\frac{2}{t}(1-\cosh at)$
107	$\arctan\frac{k}{p}$	$\frac{1}{t}\sin kt$
108	$\frac{1}{p}\arctan\frac{k}{p}$	$\text{Si}(kt)$
109	$e^{k^2p^2}erfc(kp) \quad (k>0)$	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$
110	$\frac{1}{p}e^{k^2p^2}erfc(kp) \quad (k>0)$	$erf\left(\frac{t}{2k}\right)$
111	$e^{kp}erfc\sqrt{kp} \quad (k>0)$	$\frac{\sqrt{k}}{\pi\sqrt{t}\,(t+k)}$
112	$\frac{1}{\sqrt{p}}erfc\left(\sqrt{kp}\right)$	$=0$ when $0<t<k$ $=(\pi t)^{-1/2}$ when $t>k$
113	$\frac{1}{\sqrt{p}}e^{kp}erfc\left(\sqrt{kp}\right) \quad (k>0)$	$\frac{1}{\sqrt{\pi(t+k)}}$
114	$erf\left(\frac{k}{\sqrt{p}}\right)$	$\frac{1}{\pi t}\sin\left(2k\sqrt{t}\right)$
115	$\frac{1}{\sqrt{p}}e^{\frac{k^2}{p}}erfc\left(\frac{k}{\sqrt{p}}\right)$	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$
116	$\pi e^{-kp}I_0(kp)$	$=\left[t(2k-t)\right]^{-1/2}$ when $0<t<2k$ $=0$ when $t>2k$
117	$e^{-kp}I_1(kp)$	$=\frac{k-t}{\pi k\sqrt{t(2k-t)}}$ when $0<t<2k$ $=0$ when $t>2k$

	$\overline{f}(p)$	**F(t)** $(t>0)$
118	$e^{ap}E_1(ap)$	$\frac{1}{t+a} \quad (a>0)$
119	$\frac{1}{a}-pe^{ap}E_1(ap)$	$\frac{1}{(t+a)^2} \quad (a>0)$
120	$\left(\frac{\pi}{2}-\mathrm{Si}\,p\right)\cos p+\mathrm{Ci}\,p\sin p$	$\frac{1}{t^2+1}$
121	$\left(\frac{\pi}{2}-\mathrm{Si}\,p\right)\sin p-\mathrm{Ci}\,p\cos p$	$\frac{t}{t^2+1}$

3. Laplace Transforms from Carslaw and Jaeger (1959)

Carslaw and Jaeger (1959) write $q=\sqrt{\frac{p}{\kappa}}$.

κ and x are always real and positive. α and h are unrestricted.

	***f*(*p*)**	***F*(*t*)** $(t>0)$
6	e^{-qx}	$\frac{x}{2\sqrt{\pi\kappa t^3}}e^{-\frac{x^2}{4\kappa t}}$
7	$\frac{e^{-qx}}{q}$	$\left(\frac{\kappa}{\pi t}\right)^{1/2}e^{-\frac{x^2}{4\kappa t}}$
8	$\frac{e^{-qx}}{p}$	$erfc\frac{x}{2\sqrt{\kappa t}}$
9	$\frac{e^{-qx}}{pq}$	$2\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}-x\,erfc\frac{x}{2\sqrt{\kappa t}}$
10	$\frac{e^{-qx}}{p^2}$	$\left(t+\frac{x^2}{2\kappa}\right)erfc\frac{x}{2\sqrt{\kappa t}}-x\left(\frac{t}{\pi\kappa}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}$
11	$\frac{e^{-qx}}{p^{1+\frac{1}{2}n}}$ $n=0,1,2,\ldots$	$(4t)^{\frac{1}{2}n}i^n erfc\frac{x}{2\sqrt{\kappa t}}$ $i^n erfc\{u\}=\int_u^\infty i^{n-1}erfc\{\beta\}\,d\beta$, the n^{th} repeated integral of the complementary error function with $i^0 erfc\{u\}=erfc\{u\}$
12	$\frac{e^{-qx}}{q+h}$	$\left(\frac{\kappa}{\pi t}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}-h\kappa e^{hx+\kappa th^2}erfc\left\{\frac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$
13	$\frac{e^{-qx}}{q(q+h)}$	$\kappa e^{hx+\kappa th^2}erfc\left\{\frac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$
14	$\frac{e^{-qx}}{p(q+h)}$	$\frac{1}{h}erfc\frac{x}{2\sqrt{\kappa t}}-\frac{1}{h}e^{hx+\kappa th^2}erfc\left\{\frac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$
15	$\frac{e^{-qx}}{pq(q+h)}$	$\frac{2}{h}\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}-\frac{(1+hx)}{h^2}erfc\frac{x}{2\sqrt{\kappa t}}+\frac{1}{h^2}e^{hx+\kappa th^2}$ $*erfc\left\{\frac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$

	$f(p)$	$F(t)\quad (t>0)$
16	$\dfrac{e^{-qx}}{q^{n+1}(q+h)}$	$\dfrac{\kappa}{(-h)^n}e^{hx+\kappa th^2}erfc\left\{\dfrac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}-\dfrac{\kappa}{(-h)^n}\sum_{r=0}^{n-1}\left[-2h\sqrt{\kappa t}\right]^r$ $*i^r erfc\dfrac{x}{2\sqrt{\kappa t}}$
17	$\dfrac{e^{-qx}}{(q+h)^2}$	$-2h\left(\dfrac{\kappa^3 t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}+\kappa\left(1+hx+2h^2\kappa t\right)e^{hx+\kappa th^2}$ $*erfc\left\{\dfrac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$
18	$\dfrac{e^{-qx}}{p(q+h)^2}$	$\dfrac{1}{h^2}erfc\dfrac{x}{2\sqrt{\kappa t}}-\dfrac{2}{h}\left(\dfrac{\kappa t}{\pi}\right)^{\frac{1}{2}}e^{-\frac{x^2}{4\kappa t}}-\dfrac{1}{h^2}\left(1-hx-2h^2\kappa t\right)e^{hx+\kappa th^2}$ $*erfc\left\{\dfrac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right\}$
19	$\dfrac{e^{-qx}}{p-\alpha}$	$\dfrac{1}{2}e^{\alpha t}\left\{e^{-x\sqrt{\frac{a}{\kappa}}}erfc\left[\dfrac{x}{2\sqrt{\kappa t}}-\sqrt{\alpha t}\right]+e^{x\sqrt{\frac{a}{\kappa}}}erfc\left[\dfrac{x}{2\sqrt{\kappa t}}+\sqrt{\alpha t}\right]\right\}$
20	$\dfrac{1}{p^{3/4}}e^{-qx}$	$\dfrac{1}{\pi}\left(\dfrac{x}{2t\kappa^{\frac{1}{2}}}\right)^{\frac{1}{2}}e^{-\frac{x^2}{8\kappa t}}K_{\frac{1}{4}}\left(\dfrac{x^2}{8\kappa t}\right)$
21	$\dfrac{1}{p^{1/2}}K_{2v}(qx)$	$\dfrac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{8\kappa t}}K_v\left(\dfrac{x^2}{8\kappa t}\right)$
22	$I_v(qx')K_v(qx)\quad (x>x')$ $I_v(qx)K_v(qx')\quad (x<x')$	$\dfrac{1}{2t}e^{-\left(\frac{x^2+x'^2}{4\kappa t}\right)}I_v\left(\dfrac{xx'}{2\kappa t}\right)\quad (v\geq 0)$
23	$K_0(qx)$	$\dfrac{1}{2t}e^{-\frac{x^2}{4\kappa t}}$
24	$\dfrac{1}{p}e^{x/p}$	$I_0\left[2\sqrt{xt}\right]$
25	$\dfrac{\exp\left\{xp-x\left[(p+a)(p+b)\right]^{\frac{1}{2}}\right\}}{\left[(p+a)(p+b)\right]^{\frac{1}{2}}}$	$e^{-\frac{1}{2}(a+b)(t+x)}I_0\left\{\dfrac{1}{2}(a-b)\left[t(t+2x)\right]^{\frac{1}{2}}\right\}$
26	$p^{\frac{1}{2}v-1}K_v\left(x\sqrt{p}\right)$	$x^{-v}2^{v-1}\int_{\frac{x^2}{4t}}^{\infty}e^{-u}u^{v-1}du$

	$f(p)$	$F(t)\quad (t>0)$
27	$\left[p-\sqrt{p^2-x^2}\right]^{\nu} \quad (\nu>0)$	$\dfrac{\nu x^{\nu} I_{\nu}(xt)}{t}$
28	$\dfrac{\exp\left\{x\left[(p+a)^{\frac{1}{2}}-(p+b)^{\frac{1}{2}}\right]^2\right\}}{(p+a)^{\frac{1}{2}}(p+b)^{\frac{1}{2}}\left[(p+a)^{\frac{1}{2}}+(p+b)^{\frac{1}{2}}\right]^{2\nu}} \quad (\nu>0)$	$\dfrac{t^{\frac{1}{2}\nu} e^{-\frac{1}{2}(a+b)t} I_{\nu}\left\{\frac{1}{2}(a-b)t^{\frac{1}{2}}(t+4x)^{\frac{1}{2}}\right\}}{(a-b)^{\nu}(t+4x)^{\frac{1}{2}\nu}}$
29	$\dfrac{e^{-qx}}{(p-\alpha)^2}$	$\frac{1}{2}e^{\alpha t}\left\{\left(t-\frac{x}{2\sqrt{\kappa\alpha}}\right)e^{-x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}-\sqrt{\alpha t}\right] + \left(t+\frac{x}{2\sqrt{\kappa\alpha}}\right)e^{x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}+\sqrt{\alpha t}\right]\right\}$
30	$\dfrac{e^{-qx}}{q(p-\alpha)}$	$\frac{1}{2}e^{\alpha t}\left(\frac{\kappa}{\alpha}\right)^{\frac{1}{2}}\left\{e^{-x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}-\sqrt{\alpha t}\right] - e^{x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}+\sqrt{\alpha t}\right]\right\}$
31	$\dfrac{e^{-qx}}{(p-\alpha)(q+h)} \quad (a\neq\kappa h^2)$	$\frac{1}{2}e^{\alpha t}\left\{\frac{\kappa^{\frac{1}{2}}}{h\kappa^{\frac{1}{2}}+\alpha^{\frac{1}{2}}}e^{-x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}-\sqrt{\alpha t}\right] + \frac{\kappa^{\frac{1}{2}}}{h\kappa^{\frac{1}{2}}-\alpha^{\frac{1}{2}}}\cdot e^{x\sqrt{\frac{\alpha}{\kappa}}} erfc\left[\frac{x}{2\sqrt{\kappa t}}+\sqrt{\alpha t}\right]\right\} - \frac{h\kappa}{h^2\kappa-\alpha}e^{hx+h^2\kappa t} erfc\left[\frac{x}{2\sqrt{\kappa t}}+h\sqrt{\kappa t}\right]$
32	$\dfrac{1}{p}\ln p$	$-\ln(Ct) \quad \ln C=\gamma=0.5772...$
33	$p^{\frac{1}{2}\nu}K_{\nu}\left(x\sqrt{p}\right)$	$\dfrac{x^{\nu}}{(2t)^{\nu+1}}e^{-\frac{x^2}{4t}}$

4. Laplace Transforms for well hydraulics (Hantush, 1964)

	$f(p)$	$\boldsymbol{F(t)}$ $(t>0)$
3	$\frac{1}{p(p+a)}$	$\frac{1}{a}\left[1-\exp(-at)\right]$
6	$\frac{1}{\sqrt{p}}\exp\left\{-k\sqrt{p}\right\}$	$\frac{1}{\sqrt{\pi t}}\exp\left\{-\frac{k^2}{4t}\right\}$
7	$\frac{1}{p}\exp\left\{-k\sqrt{p}\right\}$	$erfc\left(\frac{k}{\sqrt{4t}}\right)$
8	$\frac{1}{p^{1+\frac{n}{2}}}\exp\left\{-k\sqrt{p}\right\}$	$(4t)^{\frac{n}{2}}\, i^n erfc\left(\frac{k}{\sqrt{4t}}\right)$ $i^n erfc$ is n^{th} repeated integral of *erfc*
9	$K_0\left(k\sqrt{p}\right)$	$\frac{1}{2t}\exp\left\{-\frac{k^2}{4t}\right\}$
10	$\frac{1}{p}K_0\left(k\sqrt{p}\right)$	$\frac{1}{2}W\left(\frac{k^2}{4t}\right)$ *W* is the Theis well function
11	$K_0\left(k\sqrt{p+a}\right)$	$\frac{1}{2t}\exp\left\{-at-\frac{k^2}{4t}\right\}$
12	$\frac{1}{p}K_0\left(k\sqrt{p+a}\right)$	$\frac{1}{2}W\left(\frac{k^2}{4t},k\sqrt{a}\right)$ *W* is the well function for leaky aquifers (Hantush, 1964; p. 321)
13	$\frac{1}{p}K_0\left(k\sqrt{p+a\sqrt{p}}\right)$	$\frac{1}{2}H\left(\frac{k^2}{4t},\frac{ka}{4}\right)$ *H* is a well function defined by Hantush (164; p. 312)
14	$\frac{K_0\left(k\sqrt{p}\right)}{pK_0\left(k_1\sqrt{p}\right)}$	$A\left(\frac{t}{k_1^2},\frac{k}{k_1}\right)$ *A* is the flowing well function for leaky aquifers (Hantush, 1964; p. 309)
15	$\frac{K_0\left(k\sqrt{p}\right)}{p\left(k_1\sqrt{p}\right)K_1\left(k_1\sqrt{p}\right)}$	$\frac{1}{2}S\left(\frac{t}{k_1^2},\frac{k}{k_1}\right)$ *S* is the function defined by Hantush (1964; p. 318)
16	$\frac{K_0\left(k\sqrt{p+a}\right)}{pK_0\left(k_1\sqrt{p+a}\right)}$	$Z\left(\frac{t}{k_1^2},\frac{k}{k_1},k_1\sqrt{a}\right)$ *Z* is the function defined by Hantush (1964; p. 325)