THE METHOD OF FEYNMAN FORMULAE FOR DESCRIPTION OF EVOLUTION SYSTEMS

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1. INTRODUCTION

Consider an evolution equation

$$\frac{\partial f}{\partial t} = Lf.$$

Example: $f : [0, t_0] \to X = \mathcal{F}(Q)$ — a Banach space of functions on Q, Q — configuration space of the system

$$L = \frac{1}{2}\Delta \quad \Rightarrow \quad \text{heat equation,}$$

$$L = \frac{i}{2}\Delta \quad \Rightarrow \quad \text{Schrödinger equation,}$$

we consider $L : L =$ diff. or pseudo-diff. operator.

Consider an evolution equation

$$\frac{\partial f}{\partial t} = Lf.$$

If L is a bounded operator on X, then the solution of the Cauchy problem for this equation with the initial value $f(0) = f_0 \in X$ for all $f_0 \in X$ is given by

$$f(t,q) = (e^{tL} f_0)(q),$$

where

$$e^{tL} : \quad e^{tL} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n.$$

In a general case: $e^{tL} \mapsto T_t$.

Def.: One parameter family $(T_t)_{t \geqslant 0}$ of bounded operators $T_t : X \to X$ is called C_0-*semigroup*, if $T_0 = \mathrm{Id}$, $T_{s+t} = T_s \circ T_t$ for all $s, t \geqslant 0$ and $\lim_{t \to 0} \|T_t \varphi - \varphi\|_X = 0$ for all $\varphi \in X$.

Def.: If $(T_t)_{t\geqslant 0}$ is a C_0-semigroup on X then its *generator* is an operator L:

$$L\varphi := \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t}$$

with the domain

$$\mathrm{Dom}(L) := \left\{ \varphi \in X \,\middle|\, \quad \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t} \quad \text{exists as a strong limit} \right\}.$$

If L is the generator of a C_0-semigroup $(T_t)_{t\geqslant 0}$ on a Banach space X, then the (mild) solution of the Cauchy problem for the equation

$$\frac{\partial f}{\partial t} = Lf$$

with the initial value $f(0) = f_0 \in X$ for all $f_0 \in X$ is given by

$$f(t) = T_t f_0.$$

Probabilistic setting: L is the generator of a Markov process $(\xi_t)_{t\geqslant 0}$ with transitional probability $P_t(q, dy)$, i.e.

$$T_t f_0(q) = \int f_0(y) P_t(q, dy) = \mathbb{E}^q[f_0(\xi_t)].$$

To construct $(T_t \equiv e^{tL})_{t \geqslant 0}$ with the given generator L

$$\Updownarrow$$

To solve the evolution equation $\frac{\partial f}{\partial t} = Lf$

$$\Updownarrow$$

To find the transition probability $P(t, x, dy)$

If the desired semigroup is not known explicitly it can be approximated.
Our method is to approximate via the Chernoff Theorem.

This approach has been introduced in the papers

- Smolyanov, O.G., Weizsäcker, H.v., Wittich O.: Brownian Motion on a Manifold as Limit of Stepwise Conditioned Standard Brownian Motions, in: *Stochastic Proceses, Physics and Geometry: New Interplays. II: A Volume in Honor of Sergio Albeverio*, American Mathematical Society, Providence (RI) 2000, 589–602.
- Smolyanov, O.G., Tokarev, A.G., Truman A.: Hamiltonian Feynman path integrals via the Chernoff formula, *J. Math. Phys.* **43** (2002), 5161–5171.

and is actively developing now for various types of evolution equations on different geometric objects.

Rem.: Note, that a Schrödinger group e^{itL} is also a semigroup with the generator iL.

$\mathcal{L}(X)$ — the space of all continuous linear operators on X equipped with the strong topology, $\mathrm{Dom}(L)$ the domain of L.

The Chernoff theorem: *Let the mapping $F : [0, \infty) \to \mathcal{L}(X)$ be*

- *strongly continuous,*
- $F(0) = \mathrm{Id}$*,*
- $\|F(t)\| \leqslant e^{at}$ *for some $a \in \mathbb{R}$ and all $t \geqslant 0$,*
- *the operator $F'(0) \restriction_D$ for some vector subspace D of $\mathrm{Dom}(F'(0))$ admits the closure L.*

If L is a generator of a C_0-semigroup $T_t \equiv e^{tL}$ then for any $t_0 > 0$

$$e^{tL} = \lim_{n\to\infty} \left[F(t/n)\right]^n$$

where the limit is taken in the strong topology, and this convergence is uniform with respect to $t \in [0, t_0]$ for any $t_0 > 0$.

Def.: A family of operators $(F(t))_{t\geqslant 0}$ is called *Chernoff equivalent* to the semigroup $(T_t)_{t\geqslant 0}$ if this family satisfies the assertions of the Chernoff theorem w.r.t. $(T_t)_{t\geqslant 0}$, i.e.

$$F(t) \sim T_t \quad \Rightarrow \quad T_t = \lim_{n\to\infty} \big[F(t/n)\big]^n$$

Example [Trotter formula]. Let L_1, L_2, $L_1 + L_2$ be generators of C_0-semigroups e^{tL_1}, e^{tL_2} and $e^{t(L_1+L_2)}$ on a Banach space X respectively, let L_1 and L_2 do not commute. Then

$$e^{tL_1} \circ e^{tL_2} \neq e^{t(L_1+L_2)} \neq e^{tL_2} \circ e^{tL_1}.$$

But

$$e^{tL_1} \circ e^{tL_2} \sim e^{t(L_1+L_2)}, \qquad e^{tL_2} \circ e^{tL_1} \sim e^{t(L_1+L_2)},$$

and due to the Chernoff Theorem

$$e^{t(L_1+L_2)} = \lim_{n\to\infty} \Big[e^{\frac{t}{n}L_1} \circ e^{\frac{t}{n}L_2}\Big]^n = \lim_{n\to\infty} \Big[e^{\frac{t}{n}L_2} \circ e^{\frac{t}{n}L_1}\Big]^n.$$

Rem.: The Post–Widder Inversion formula also follows from the Chernoff theorem:

$$e^{tL} = \lim_{n\to\infty}\left(\mathrm{Id} - \frac{t}{n}L\right)^{-n} \equiv \lim_{n\to\infty}\left[\frac{n}{t}R\Big(n/t, L\Big)\right]^{n}.$$

In many cases the operators $F(t)$ are integral operators, i.e.

$$T_t = \lim_{n\to\infty}\big[F(t/n)\big]^n = \lim_{N(n)\to\infty}\underbrace{\int\ldots\int}_{N(n)\,times}\ldots\ldots\ldots$$

Def.: A *Feynman formula* is a representation of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup solving the problem) by a limit of N-fold iterated integrals as $N\to\infty$.

In all cases the identity $T_t = \lim_{n\to\infty}\big[F(t/n)\big]^n$ is called Feynman formula.

Lagrangian Feynman formula:

► $F(t)$ are integral operators with elementary kernels.

► Such Feynman formula corresponds to a functional integral over a set of paths in the configuration space.

Hamiltonian Feynman formula:

► $F(t)$ are ΨDOs.

► Such Feynman formula corresponds to a functional integral over a set of paths in the phase space.

There are other kinds of Feynman formulae.

Advantages of Feynman formulae:

► Possible to consider evolutionary equations with wide variety of operators L in the r.h.s. (e.g., with $L = A + B + C$, with non-local operators); initial-boundary value problems; evolutionary equations on different geometric structures (Riemannian manifolds, graphs, infinite dimensional spaces, p-adic spaces e.t.c.).

► In many cases it is possible to get the representation of the semigroup by the limit of iterated integrals of ELEMENTARY FUNCTIONS ONLY!!! This is useful for direct computations and computer modeling of the evolution, simulation of stochastic processes, etc.

► Feynman formulae allow to obtain new Feynman–Kac formulae, to define some new functional integrals (Feynman path integrals), to investigate relations between different functional integrals, to calculate path integrals numerically.

2. HOW THE METHOD WORKS

2.1. Feynman formula for additive perturbations

Theorem: *Let* $(e^{tL_k})_{t\geqslant 0}$, $k=1,\dots,m$, *be* C_0*-semigroups on* X *and* $(F_k(t))_{t\geqslant 0}$, $k=1,\dots,m:$

$$F_k(t)\sim e^{tL_k}.$$

Let $e^{t(L_1+\dots+L_m)}$ *be a* C_0*-semigroup on* X. *Then*

$$F_1(t)\circ\dots\circ F_m(t)\sim e^{t(L_1+\dots+L_m)}$$

i.e. the Feynman formula

$$e^{t(L_1+\dots+L_m)}=\lim_{n\to\infty}\big[F_1(t/n)\circ\dots\circ F_m(t/n)\big]^n,$$

is valid (the limit is in strong topology locally uniformly w.r.t. $t\geqslant 0$*).*

Example: Let $X = C_\infty(\mathbb{R}^d)$, $b(\cdot) \in C_b(\mathbb{R}^d, \mathbb{R}^d)$, $c(\cdot) \in C_b(\mathbb{R}^d)$ and

$$L := \frac{1}{2}\Delta + b\nabla + c$$

Take

$$\begin{cases} F_1(t) = e^{tc}, \\ F_2(t): \quad F_2(t)\varphi(q) = \varphi(q + tb(q)) \quad \Rightarrow \quad F_2(t) \sim e^{tb\nabla}, \\ F_3(t) = e^{\frac{t}{2}\Delta}: \quad e^{\frac{t}{2}\Delta}\varphi(q) = (2\pi t)^{(-d/2)} \int\limits_{\mathbb{R}^d} e^{-\frac{|q-y|^2}{2t}} \varphi(y)dy. \end{cases}$$

Then

$$F(t) \equiv F_1(t) \circ F_2(t) \circ F_3(t) \sim e^{t(\frac{1}{2}\Delta + b\nabla + c)}.$$

Therefore, using this family $(F(t))_{t \geqslant 0}$,

$$F(t)\varphi(q) = \frac{\exp(tc(q))}{\sqrt{(2\pi t)^d}} \int\limits_{\mathbb{R}^d} \exp\left(-\frac{|q + tb(q) - y|^2}{2t}\right) \varphi(y)dy,$$

we obtain the Lagrangian Feynman formula for the semigroup $e^{t(\frac{1}{2}\Delta + b\nabla + c)}$:

$$e^{t(\frac{1}{2}\Delta+b\nabla+c)}\varphi(q_0) = \lim_{n\to\infty}\left(\left[F(t/n)\right]^n\varphi\right)(q_0)$$

$$= \lim_{n\to\infty}(2\pi t/n)^{(-dn/2)}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} e^{\frac{t}{n}\sum\limits_{k=1}^{n}c(q_{k-1})}e^{-\sum\limits_{k=1}^{n}\frac{|q_{k-1}+b(q_{k-1})t/n-q_k|^2}{2t/n}}\times$$

$$\times\,\varphi(q_n)dq_1\cdots dq_n =$$

rearranging the expression in the integrand we have

$$= \lim_{n\to\infty}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} e^{\frac{t}{n}\sum_{k=1}^{n}c(q_{k-1})}e^{-\sum_{k=1}^{n}b(q_{k-1})\cdot(q_{k-1}-q_k)}\times$$

$$\times\, e^{-\frac{t}{2n}\sum_{k=1}^{n}|b(q_{k-1})|^2}p^{BM}_{t/n}(q_0-q_1)\cdots p^{BM}_{t/n}(q_{n-1}-q_n)\varphi(q_n)dq_1\cdots dq_n =$$

$$= \mathbb{E}^{q_0}\left[e^{\int\limits_0^t c(\xi_s)ds}e^{\int\limits_0^t b(\xi_s)\cdot d\xi_s}e^{-\frac{1}{2}\int\limits_0^t|b(\xi_s)|^2ds}\varphi(\xi_t)\right].$$

with $p_t^{BM}(x) = (2\pi t)^{(-d/2)}\exp\left\{-\frac{|x|^2}{2t}\right\}$ and Brownian motion ξ_t.

In the similar way we obtain a Feynman formula for the Schrödinger type equation: let $X = L_2(\mathbb{R}^d)$, $b \in \mathbb{R}^d$, $c(\cdot) \in C_b(\mathbb{R}^d)$ and

$$L = \frac{i}{2}\Delta - b\nabla - ic.$$

Then the family $(F(t))_{t\geqslant 0}$,

$$F(t)\varphi(q) = \frac{\exp(-itc(q))}{\sqrt{(2\pi it)^d}} \int\limits_{\mathbb{R}^d} \exp\left(\frac{i|q + tb(q) - y|^2}{2t}\right)\varphi(y)dy,$$

is Chernoff equivalent to the semigroup e^{tL} and

$$e^{tL}\varphi(q_0) = \lim_{n\to\infty} \left[e^{-i\frac{t}{n}c} \circ e^{-\frac{t}{n}b\nabla} \circ e^{\frac{i}{2}\Delta}\right]^n \varphi(q_0) =$$

$$= \lim_{n\to\infty} (2\pi it)^{(-dn/2)} \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} e^{-i\frac{t}{n}\sum\limits_{k=1}^{n} c(q_{k-1})} e^{i\sum\limits_{k=1}^{n} \frac{|q_k - q_{k-1} + \frac{t}{n}b|^2}{2t/n}} \varphi(q_n)dq_1 \dots dq_n =$$

$$= \int e^{-i\int\limits_0^t [c(\xi(s)) - \frac{1}{2}|b|^2]ds} e^{i\int\limits_0^t b\cdot d\xi(s)} \varphi(\xi(t))\Phi^{q_0}(d\xi)$$

2.2. Feynman formula for multiplicative perturbations

Let Q be a metric space, $X = C_b(Q)$ or $X = C_\infty(Q)$, $\|f\|_X = \sup\limits_{q\in Q} |f(q)|$.

Consider a C_0-semigroup $(e^{tL})_{t\geqslant 0}$ on X and a function

$$a(\cdot) : Q \to [c_1, c_2] \subset (0, \infty), \quad 0 < c_1 < c_2 < +\infty, \quad a(\cdot) \in C(Q).$$

Define $\widetilde{L}$: $\quad \widetilde{L}\varphi(q) = a(q)(L\varphi)(q), \quad \mathrm{Dom}(\widetilde{L}) = \mathrm{Dom}(L).$

Theorem: *Let* $F(t) \sim e^{tL}$. *Consider a family of operators* $(\widetilde{F}(t))_{t\geqslant 0}$ *defined by the formula*

$$\widetilde{F}(t)\varphi(q) = (F(a(q)t)\varphi)(q).$$

Then the family $(\widetilde{F}(t))_{t\geqslant 0}$ *acts in* X *and* $\widetilde{F}(t) \sim e^{t\widetilde{L}}$. *Therefore, the Feynman formula*

$$e^{t\widetilde{L}} = \lim_{n\to\infty} \left[\widetilde{F}(t/n)\right]^n$$

is valid locally uniformly with respect to $t \geqslant 0$ *in* $\mathcal{L}(X)$.

Cor.: Consider a Markov process $(\xi_t)_{t\geqslant 0}$ with the state space Q, transition density $P_t(q,dy)$ and generator L. Let the corresponding semigroup $(e^{tL})_{t\geqslant 0}$,

$$e^{tL}\varphi(q) = \mathbb{E}^q\left[\varphi(\xi_t)\right] \equiv \int\limits_Q \varphi(y)P_t(q,dy),$$

be a C_0-semigroup on a Banach space X, where $X = C_b(Q)$ or $X = C_\infty(Q)$. Then the family $(\widetilde{F}(t))_{t\geqslant 0}$ defined by the formula

$$\widetilde{F}_t\varphi(q) = \int\limits_Q \varphi(y)P_{a(q)t}(q,dy),$$

is Chernoff equivalent to the semigroup $(e^{t\widetilde{L}})_{t\geqslant 0}$. Therefore, the following Lagrangian Feynman formula is valid for any $\varphi\in X$ and any $q_0\in Q$:

$$e^{t\widetilde{L}}\varphi(q_0) = \lim_{n\to\infty}\int\limits_Q\cdots\int\limits_Q \varphi(q_n)P_{a(q_0)t/n}(q_0,dq_1)P_{a(q_1)t/n}(q_1,dq_2)\cdots$$

$$\cdots P_{a(q_{n-1})t/n}(q_{n-1},dq_n).$$

Rem.: A process $\widetilde{\xi}_t$, whose semigroup is $e^{t\widetilde{L}}$, can be obtained from ξ_t by some random time change.

Note that $\widetilde{\xi}_t \neq \xi_{a(q)t}$ and $P_t^{\widetilde{\xi}}(q,dy) \neq P_{a(q)t}^{\xi}(q,dy)$!

Rem.: Previous results are generalized for the Gaussian case:

$$L: \quad L\varphi(q) = \mathrm{tr}(A(q)\,\mathrm{Hess}\varphi(q)) + b(q)\cdot\nabla\varphi(q) + c(q)\varphi(q),$$

$$F(t): \quad F(t)\varphi(q) = \frac{\exp(tc(q))}{\sqrt{\det A(q)(2\pi t)^d}} \times$$

$$\times \int\limits_{\mathbb{R}^d} \exp\left(\frac{-A^{-1}(q)(q-y+tb(q))\cdot(q-y+tb(q))}{2t}\right)\varphi(y)dy,$$

$$F(t) \sim e^{tL}.$$

2.3. Feynman formula for the Cauchy–Dirichlet problem

G – bounded domain in $\mathbb{R}^d$ and $X = C_\infty(\mathbb{R}^d)$ or
G – bounded domain in a Riemannian manifold K and $X = C_b(K)$,
L – dif. operator, e^{tL} – C_0-semigroup on X. Consider

$$\begin{cases} \frac{\partial f}{\partial t}(t,q) = Lf(t,q), \quad t > 0, q \in G, \\ f(0,q) = f_0(q), \quad q \in G, \\ f(t,q) = 0, \quad q \in \partial G. \end{cases}$$

Let $Y = \{\varphi \in C(G), \lim\limits_{q \to \partial G} \varphi(q) = 0\}$ and $\|f\|_Y = \sup_{x \in G} |f(x)|$.

Let $\exists$ a C_0-semigroup $(e^{tL_o})_{t \geqslant 0}$ on Y such that for all $f_0 \in Y$ and $q \in G$

$$f(t,q) = e^{tL_o} f_0(q).$$

Theorem: *Let $F(t) \sim e^{tL}$ in X. Consider $(F_o(t))_{t \geqslant 0}$ in Y:*

$$F_o(t)\varphi(q) = \psi_t(q)[F(t)\varphi](q),$$

where $\psi_t \in C_c^\infty(G)$ and $\psi_t(q) \to \mathbb{I}_G(q)$ as $t \to 0$. Then

$$F_o(t) \sim e^{tL_o}$$

and, hence,

$$e^{tL_o} = \lim_{n\to\infty} \left[F_o(t/n)\right]^n.$$

Example: Consider an operator L:

$$L\varphi(q) = \mathrm{tr}(A(q)\,\mathrm{Hess}\varphi(q)) + b(q)\cdot\nabla\varphi(q) + c(q)\varphi(q).$$

As before $F(t) \sim e^{tL}$ with

$$F(t): \quad F(t)\varphi(q) = \frac{\exp(tc(q))}{\sqrt{\det A(q)(2\pi t)^d}} \times$$

$$\times \int_{\mathbb{R}^d} \exp\left(\frac{-A^{-1}(q)(q-y+tb(q))\cdot(q-y+tb(q))}{2t}\right)\varphi(y)dy.$$

Then for all $q_0 \in G$

$$f(t,q_0) = e^{tL_o} f_0(q_0) = \mathbb{E}^{q_0}\left[\exp\left(\int_0^t c(\xi_\tau)d\tau\right) f_0(\xi_t) \,\middle|\, t < \tau_G\right] =$$

$$= \lim_{n\to\infty} \int_G \cdots \int_G \exp\left(-\sum_{k=1}^n A^{-1}(q_{k-1})b(q_{k-1})\cdot(q_{k-1}-q_k)\right)\times$$

$$\times \exp\left(\frac{t}{n}\sum_{k=1}^n c(q_{k-1})\right)\exp\left(-\frac{t}{2n}\sum_{k=1}^n A^{-1}(q_{k-1})b(q_{k-1})\cdot b(q_{k-1})\right)\times$$

$$\times f_0(q_n) p_A(t/n, q_0, q_1)\cdots p_A(t/n, q_{n-1}, q_n)\, dq_1 \dots dq_n,$$

where

$$p_A(t,x,y) := \frac{1}{\sqrt{\det A(x)(2\pi t)^d}} \exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{2t}\right).$$

2.4. Feynman formulae for the evolution in a Riemannian manifold

Consider a compact m-dimensional Riemannian manifold K and the Cauchy problem

$$\frac{\partial f}{\partial t}(t,q) = -\frac{1}{2}\Delta_K f(t,q) + c(q) f(t,q),$$
$$f(0,q) = f_0(q),$$

where $\Delta_K = -\mathrm{tr}\nabla^2$ — Laplace–Beltrami operator. Again the solution:

$$f(t,q) = \mathbb{E}^q\left[e^{\int_0^t c(\xi(\tau))d\tau} f_0(\xi(t))\right] =$$

$$= \lim_{n\to\infty} \int\limits_K \dots \int\limits_K e^{\frac{t}{n}\sum\limits_{k=1}^{n} c(q_{k-1})} p^{BM}(t/n, q, q_1) p^{BM}(t/n, q_1, q_2) \dots$$
$$\dots p^{BM}(t/n, q_{n-1}, q_n) f_0(q_n) \mathrm{vol}_K(dq_1) \dots \mathrm{vol}_K(dq_n)$$

But the function $p^{BM}(t,x,y)$ is not elementary any more!

Denote by vol_K the Borel measure on K and by ρ - a distance in K, generated by Riemannian structure of K. Due to Nash theorem we can and shall assume that K is a smooth m-dimensional manifold isometrically embedded into a Euclidean space $\mathbb{R}^N$ and $\Phi : K \to \mathbb{R}^N$ is a smooth embedding.

Define the following functions:

$$g^E(t,x,z) = \frac{1}{(2\pi t)^{m/2}} e^{-\frac{||\Phi(x)-\Phi(z)||^2_{\mathbb{R}^N}}{2t}}, \quad t>0,\ x,z \in K,$$

$$g^I(t,x,z) = \frac{1}{(2\pi t)^{m/2}} e^{-\frac{\rho(x,z)^2}{2t}}, \quad t>0,\ x,z \in K.$$

$$p^E(t,x,z) = \frac{g^E(t,x,z)}{\int_K g^E(t,x,z)\mathrm{vol}_K(dz)},\ t>0, \quad x,z \in K,$$

$$p^I(t,x,z) = \frac{g^I(t,x,z)}{\int_K g^I(t,x,z)\mathrm{vol}_K(dz)}, \quad t>0,\ x,z \in K.$$

Let $\mathrm{scal}(x) \equiv \mathrm{tr}\,\mathrm{Ricci}(x)$ be a scalar curvature of the manifold K at the point $x \in K$. Denote by $r^2(x)$ the square of the norm of the vector-valued mean curvature of the manifold at the point x. We assume that functions $\mathrm{scal}(\cdot)$ and $r^2(\cdot)$ - are continuous on K.

Proposition: The following operator families are Chernoff equivalent to the semigroup $e^{-\frac{t}{2}\Delta_K + tc}$:

$$F_p^E(t): \quad (F_p^E(t)f)(x) = \int_K e^{tc(x)} f(z) p^E(t,x,z) \mathrm{vol}_K(dz),$$

$$F_p^I(t): \quad (F_p^I(t)f)(x) = \int_K e^{tc(x)} f(z) p^I(t,x,z) \mathrm{vol}_K(dz),$$

$$F_g^E(t): \quad (F_g^E(t)f)(x) = \int_K e^{tc(x)} e^{\frac{t}{4}\mathrm{scal}(x)} e^{-\frac{t}{8}r^2(x)} f(z) g^E(t,x,z) \mathrm{vol}_K(dz)$$

$$F_g^I(t): \quad (F_g^I(t)f)(x) = \int_K e^{tc(x)} e^{\frac{t}{6}\mathrm{scal}(x)} f(z) g^I(t,x,z) \mathrm{vol}_K(dz)$$

2.5. Hamiltonian Feynman formula for Feller semigroups

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$. Define a ΨDO $\widehat{H}(\cdot, D)$ with the symbol $H(q,p)$ by

$$\widehat{H}(q,D)\phi(q) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q_1)} H(q,p)\phi(q_1)\, dq_1\, dp.$$

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be measurable, locally bounded in both variables (q,p) and $H(q,\cdot)$ is a CNDF for each $q \in \mathbb{R}^d$, i.e.

$$H(q,p) = c(q) + ib(q)\cdot p + p\cdot A(q)p + \int_{y\neq 0} \left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1+|y|^2}\right) N(q,dy).$$

Let

$$\begin{cases} \sup_{q\in\mathbb{R}^d} |H(q,p)| \le \kappa(1+|p|^2) & \text{for all} \quad p \in \mathbb{R}^d \quad \text{and some} \quad \kappa > 0 \\ p \mapsto H(q,p) & \text{is uniformly (w.r.t.} \quad q \in \mathbb{R}^d\text{) continuous at} \quad p = 0, \\ q \mapsto H(q,p) & \text{is continuous for all} \quad p \in \mathbb{R}^d. \end{cases}$$

Let $\exists$ a C_0-semigroup $(e^{-t\widehat{H}})_{t\geqslant 0}$ on $C_\infty(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d)$ be a core for $\widehat{H}$.

Note that $e^{-t\widehat{H}} \neq \widehat{e^{-tH}}$!

Theorem: *Let* $F(t) := \widehat{e^{-tH}}$ *— a* ΨDO *with the the symbol* $e^{-tH(q,p)}$*, i.e. for each* $\phi \in C_c^\infty(\mathbb{R}^d)$

$$F(t)\phi(q) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}e^{-tH(q,p)}\phi(q_1)\,dq_1\,dp.$$

Then

$$F(t) = \widehat{e^{-tH}} \sim e^{-t\widehat{H}}$$

and, hence, the Feynman formula

$$e^{-t\widehat{H}} = \lim_{n\to\infty}\left[F(t/n)\right]^n$$

is valid in $\mathcal{L}(C_\infty(\mathbb{R}^d))$ *locally uniformly with respect to* $t \geqslant 0$.

Corollary:

$$(e^{-t\widehat{H}}\phi)(q_0) = \mathbb{E}^{q_0}[\phi(\xi_t)] =$$

$$= \lim_{n\to\infty} (2\pi)^{-dn} \int\limits_{(\mathbb{R}^d)^{2n}} \exp\left(i \sum_{k=1}^{n} p_k \cdot (q_k - q_{k-1}) \right) \times$$

$$\times \exp\left(-\frac{t}{n} \sum_{k=1}^{n} H(q_{k-1}, p_k) \right) \phi(q_n)\, dq_1\, dp_1 \cdots dq_n\, dp_n$$

$$= \int\limits_{E_t^{q_0}} e^{-\int\limits_0^t H(q(s),p(s))ds} \varphi(q(t)) \Phi_{q_0}(dqdp).$$

Here ξ_t is a Feller process with the generator $-\widehat{H}$, and Φ_{q_0} is a Feynman pseudomeasure on a set $E_t^{q_0}$ of paths in the phase space of a system.

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THANKS FOR THE ATTENTION!